

Analysis of Minimizers of the Lawrence-Doniach Energy for Superconductors in Applied Fields

Patricia Bauman* and Guanying Peng*

*Department of Mathematics, Purdue University
West Lafayette, IN 47907, USA
E-mail address: bauman@math.purdue.edu
E-mail address: gpeng@math.purdue.edu*

Abstract

We analyze minimizers of the Lawrence-Doniach energy for layered superconductors occupying a bounded generalized cylinder, $\Omega \times [0, L]$, in \mathbb{R}^3 , where Ω is a bounded simply connected Lipschitz domain in \mathbb{R}^2 . For an applied magnetic field $\vec{H}_{ex} = h_{ex}\vec{e}_3$ that is perpendicular to the layers with $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ as $\epsilon \rightarrow 0$, where ϵ is the reciprocal of the Ginzburg-Landau parameter, we prove an asymptotic formula for the minimum Lawrence-Doniach energy as ϵ and the interlayer distance s tend to zero. Under appropriate assumptions on s versus ϵ , we establish comparison results between the minimum Lawrence-Doniach energy and the minimum three-dimensional anisotropic Ginzburg-Landau energy. As a consequence, our asymptotic formula also describes the minimum three-dimensional anisotropic energy as ϵ tends to zero.

1 Introduction

The Lawrence-Doniach model was formulated by Lawrence and Doniach in 1971 as a macroscopic model for layered superconductors. While the standard Ginzburg-Landau model has been well accepted as a macroscopic model for isotropic superconductors, it does not account for the anisotropy in three-dimensional high temperature superconducting materials. For these materials, depending on the nature of the anisotropy in the material, physicists have used the Lawrence-Doniach model (which treats the superconducting material as a stack of parallel superconducting layers with nonlinear Josephson coupling between them) or the three-dimensional anisotropic Ginzburg-Landau model (which is a slight modification of the standard three-dimensional Ginzburg-Landau model).

The standard two-dimensional Ginzburg-Landau energy model (with energy given by (1.6)) has been intensively investigated. In this case, an analysis of the behavior of energy minimizers and their vortex structure in a perpendicular applied magnetic field with modulus h_{ex} in different regimes (e.g., $h_{ex} \sim |\ln \epsilon|$, $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$, or $h_{ex} \geq \frac{C}{\epsilon^2}$) is now well understood. (See [9].) However, for the Lawrence-Doniach energy (see (1.1)), an analysis for h_{ex} in the first two regimes has been done only for the gauge-periodic problem, in which the superconductor is assumed to occupy all of \mathbb{R}^3 and the gauge invariant quantities are assumed to be periodic with respect to a given parallelepiped. (See [1].) In the last regime, $h_{ex} \geq \frac{C}{\epsilon^2}$, it was shown in [2] that if C is sufficiently large, all minimizers of the Lawrence-Doniach energy are in the normal (nonsuperconducting) phase, that is, the order parameters on the layers, $\{u_n\}_{n=0}^N$, are all identically equal to zero, and the induced magnetic field, $\nabla \times \vec{A}$, is

*The authors were supported in part by NSF Grant DMS-1109459.

identically equal to the applied magnetic field. A similar result is known for the two-dimensional Ginzburg-Landau energy. (See [6].)

In this paper, we analyze the Lawrence-Doniach model in the second regime without imposing gauge periodicity assumptions. In two-dimensional superconductors, this regime for h_{ex} corresponds to a mixed state in which superconducting states and normal states (in the form of isolated vortices, i.e., zeros of u_n) coexist.

The Lawrence-Doniach model describes a layered superconductor occupying a cylinder $D = \Omega \times [0, L]$ with cross-section Ω and $N + 1$ equally spaced layers of material occupying $\Omega_n \equiv \Omega \times \{ns\}$, where Ω is a bounded simply connected Lipschitz domain in \mathbb{R}^2 and $s = \frac{L}{N}$ is the interlayer distance. Assuming an applied magnetic field $h_{ex}\vec{e}_3$, the Lawrence-Doniach energy is given by

$$\begin{aligned} \mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) &= s \sum_{n=0}^N \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} \\ &\quad + s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \vec{A} - h_{ex}\vec{e}_3|^2 dx \end{aligned} \quad (1.1)$$

for $(\{u_n\}_{n=0}^N, \vec{A})$ such that

$$\begin{cases} \{u_n\}_{n=0}^N \in [H^1(\Omega; \mathbb{C})]^{N+1} \text{ and} \\ \vec{A} \in E \equiv \{\vec{C} \in H_{loc}^1(\mathbb{R}^3; \mathbb{R}^3) : (\nabla \times \vec{C}) - h_{ex}\vec{e}_3 \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}. \end{cases} \quad (1.2)$$

Here $\epsilon > 0$ is the reciprocal of the Ginzburg-Landau parameter and $\lambda > 0$ represents the Josephson penetration depth, which is assumed to be fixed throughout this study. The applied magnetic field $h_{ex}\vec{e}_3$ is assumed to satisfy $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ as $\epsilon \rightarrow 0$. The complex valued function u_n defined in Ω is the order parameter for the n th layer and $|u_n(x_1, x_2)|^2$ is the density of superconducting electron pairs at each point (x_1, x_2, ns) on the n th layer. For a minimizer of the Lawrence-Doniach energy (1.1), $|u_n(x_1, x_2)| \sim 1$ corresponds to a superconducting state at (x_1, x_2, ns) , whereas $|u_n(x_1, x_2)| = 0$ corresponds to a normal (nonsuperconducting) state at (x_1, x_2, ns) , in which the density of superconducting electrons is zero. The vector field $\vec{A} = (A^1, A^2, A^3)$ defined on \mathbb{R}^3 is called the magnetic potential; its curl, $\nabla \times \vec{A}$, is the induced magnetic field. We let $x = (x_1, x_2, x_3)$, $\hat{\nabla} = (\partial_1, \partial_2)$, $\hat{x} = (x_1, x_2)$, $\hat{A} = (A^1, A^2)$ and $\hat{A}_n(\hat{x}) = (A^1(\hat{x}, ns), A^2(\hat{x}, ns))$, the trace of \vec{A} on the n th layer. We set $\hat{\nabla}_{\hat{A}_n} u_n = \hat{\nabla} u_n - i\hat{A}_n u_n$ on Ω . In the following, given two complex numbers u and v , we let $(u, v) = \frac{1}{2}(\bar{u}v + u\bar{v}) = \Re(u\bar{v})$, which is an inner product of $u = u_1 + iu_2$ and $v = v_1 + iv_2$ in \mathbb{C} that agrees with the inner product of (u_1, u_2) and (v_1, v_2) in \mathbb{R}^2 .

Since $\vec{A} \in H_{loc}^1(\mathbb{R}^3; \mathbb{R}^3)$, it follows from the trace theorem and the Sobolev imbedding theorem that its trace $\hat{A}_n \in H_{loc}^{\frac{1}{2}}(\mathbb{R}^2; \mathbb{R}^2) \subset L_{loc}^4(\mathbb{R}^2; \mathbb{R}^2)$ and therefore the Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A})$ is well-defined and finite. The existence of minimizers in $[H^1(\Omega; \mathbb{C})]^{N+1} \times E$ was shown by Chapman, Du and Gunzburger in [3]. Each minimizer of $\mathcal{G}_{LD}^{\epsilon,s}$ corresponds to a physically realistic state for the layered superconductor. The minimizer satisfies the Euler-Lagrange equations associated to the Lawrence-Doniach energy. This system of equations is called the Lawrence-Doniach system and it is given by

$$\begin{cases} (\hat{\nabla} - i\hat{A}_n)^2 u_n + \frac{1}{\epsilon^2} (1 - |u_n|^2) u_n + P_n = 0 & \text{on } \Omega, \\ \nabla \times (\nabla \times \vec{A}) = (j_1, j_2, j_3) & \text{in } \mathbb{R}^3, \\ (\hat{\nabla} - i\hat{A}_n) u_n \cdot \vec{n} = 0 & \text{on } \partial\Omega, \\ \nabla \times \vec{A} - h_{ex}\vec{e}_3 \in L^2(\mathbb{R}^3; \mathbb{R}^3) \end{cases}$$

for all $n = 0, 1, \dots, N$, where

$$P_n = \begin{cases} \frac{1}{\lambda^2 s^2} (u_1 \tilde{\Upsilon}_0^1 - u_0) & \text{if } n = 0, \\ \frac{1}{\lambda^2 s^2} (u_{n+1} \tilde{\Upsilon}_n^{n+1} + u_{n-1} \Upsilon_{n-1}^n - 2u_n) & \text{if } 0 < n < N, \\ \frac{1}{\lambda^2 s^2} (u_{N-1} \Upsilon_{N-1}^N - u_N) & \text{if } n = N, \end{cases}$$

$$\Upsilon_n^{n+1} = e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3} \text{ for } n = 0, 1, \dots, N-1,$$

$$j_i = -s \sum_{n=0}^N (\partial_i u_n - \imath A_n^i u_n, -\imath u_n) \chi_\Omega(x_1, x_2) dx_1 dx_2 \delta_{ns}(x_3) \text{ for } i = 1, 2,$$

$$j_3 = s \sum_{n=0}^{N-1} \frac{1}{\lambda^2 s^2} (u_{n+1} - u_n \Upsilon_n^{n+1}, \imath u_n \Upsilon_n^{n+1}) \chi_\Omega(x_1, x_2) \chi_{[ns, (n+1)s]}(x_3).$$

It was proved in [2] that a minimizer $(\{u_n\}_{n=0}^N, \vec{A})$ of (1.1) satisfies $|u_n| \leq 1$ a.e. in Ω for all $n = 0, 1, \dots, N$.

Two configurations $(\{u_n\}_{n=0}^N, \vec{A})$ and $(\{v_n\}_{n=0}^N, \vec{B})$ in $[H^1(\Omega; \mathbb{C})]^{N+1} \times E$ are called gauge equivalent if there exists a function $g \in H_{loc}^2(\mathbb{R}^3)$ such that

$$\begin{cases} u_n(\hat{x}) = v_n(\hat{x}) e^{\imath g(\hat{x}, ns)} & \text{in } \Omega, \\ \vec{A} = \vec{B} + \nabla g & \text{in } \mathbb{R}^3. \end{cases} \quad (1.3)$$

Simple calculations show that $\mathcal{G}_{LD}^{\epsilon, s}$ (and each term in $\mathcal{G}_{LD}^{\epsilon, s}$) is invariant under the above gauge transformation, i.e., for two configurations $(\{u_n\}_{n=0}^N, \vec{A})$ and $(\{v_n\}_{n=0}^N, \vec{B})$ that are related by (1.3), we have $\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) = \mathcal{G}_{LD}^{\epsilon, s}(\{v_n\}_{n=0}^N, \vec{B})$. Let $\vec{a} = \vec{a}(x)$ be any fixed smooth vector field on \mathbb{R}^3 such that $\nabla \times \vec{a} = \vec{e}_3$ in \mathbb{R}^3 . For example, we may choose $\vec{a}(x) = (0, x_1, 0)$. It was also proved in [2] that every pair $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times E$ is gauge equivalent to another pair $(\{v_n\}_{n=0}^N, \vec{B}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ where

$$K = \{\vec{C} \in E : \nabla \cdot \vec{C} = 0 \text{ and } \vec{C} - h_{ex} \vec{a} \in \check{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3; \mathbb{R}^3)\}. \quad (1.4)$$

Here the space $\check{H}^1(\mathbb{R}^3)$ represents the completion of $C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with respect to the seminorm

$$\|\vec{C}\|_{\check{H}^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla \vec{C}|^2 dx \right)^{\frac{1}{2}}.$$

In particular, any minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$ in the admissible space $[H^1(\Omega; \mathbb{C})]^{N+1} \times E$ is gauge-equivalent to a minimizer in the space $[H^1(\Omega; \mathbb{C})]^{N+1} \times K$, called the ‘‘Coulomb gauge’’ for $\mathcal{G}_{LD}^{\epsilon, s}$. It was shown in [2] that minimizers in the Coulomb gauge satisfy $u_n \in C^\infty(\Omega)$ and $\vec{A}_n \in H_{loc}^1(\mathbb{R}^2)$ for all $n = 0, 1, \dots, N$. Throughout this paper, we take $\vec{a}(x) = (0, x_1, 0)$.

Given the above definitions, our main results are the following:

Theorem 1. *Assume $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then denoting the volume of D by $|D|$, we have*

$$\begin{aligned} \left| \mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) - \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} \right| &\leq (Cs^{\frac{1}{7}} + o_\epsilon(1)) \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} \\ &= o_{\epsilon, s}(1) \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} \end{aligned}$$

as $(\epsilon, s) \rightarrow (0, 0)$ for some constant C independent of ϵ and s . In particular,

$$\lim_{(\epsilon, s) \rightarrow (0, 0)} \frac{\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A})}{h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}} = \frac{|D|}{2}.$$

(See Theorem 3.1 and Theorem 4.2.)

Here $o_\epsilon(1)$ denotes a quantity that converges to 0 as $\epsilon \rightarrow 0$ and $o_{\epsilon, s}(1)$ denotes a quantity that converges to 0 as $(\epsilon, s) \rightarrow (0, 0)$. This theorem generalizes a result in the gauge periodic case studied by Alama, Bronsard and Sandier for the energy (1.1) in which the domain Ω is replaced by a parallelogram P in \mathbb{R}^2 , the integral of $|\nabla \times \vec{A} - h_{ex} \vec{e}_3|^2$ is taken over $P \times [0, L]$ instead of over \mathbb{R}^3 , and the minimization takes place among gauge periodic configurations $(\{u_n\}_{n=0}^N, \vec{A})$ in \mathbb{R}^3 with period $P \times [0, L]$. (See [1].) In that case, they further showed that for a minimizer of the gauge periodic problem, the order parameters u_n are all equal and A^3 is identically zero. In particular, the Josephson coupling term

$$s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} \quad (1.5)$$

vanishes. They also proved that $\hat{A}(\hat{x}, \cdot)$ is periodic in x_3 with period s and established certain symmetries between the layers in \hat{A} .

In the gauge periodic case, the results of Alama, Bronsard and Sandier indicate a close connection between the Lawrence-Doniach energy and the two-dimensional Ginzburg-Landau energy GL_ϵ given by

$$GL_\epsilon(u, \hat{A}) = \frac{1}{2} \int_{\Omega} \left[|\hat{\nabla}_{\hat{A}} u|^2 + \frac{1}{2\epsilon^2} (1 - |u|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_{\mathbb{R}^2} (\text{curl} \hat{A} - h_{ex})^2 d\hat{x} \quad (1.6)$$

for h_{ex} as assumed above where curl denotes the two-dimensional curl defined by $\text{curl}(B^1, B^2) = \partial_1 B^2 - \partial_2 B^1$. We remark that for a minimizer of the two-dimensional energy GL_ϵ , the magnetic potential \hat{A} satisfies $\text{curl} \hat{A} = h_{ex}$ in $\mathbb{R}^2 \setminus \Omega$. (See Lemma 2.1 in [6].) Therefore the minimum of GL_ϵ is equal to the minimum of F_ϵ given by

$$F_\epsilon(u, \hat{A}) = \frac{1}{2} \int_{\Omega} \left[|\hat{\nabla}_{\hat{A}} u|^2 + \frac{1}{2\epsilon^2} (1 - |u|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_{\Omega} (\text{curl} \hat{A} - h_{ex})^2 d\hat{x}.$$

(See Prop. 3.4 in [9] for bounded simply connected smooth domains and Prop. 2.1 in this paper for bounded simply connected Lipschitz domains.) The main idea in our proof of the upper bound in Theorem 1 is to construct a test function that is an extension of $N+1$ copies in each layer, $\Omega \times \{ns\}$, of a two-dimensional configuration $(u, (A_1, A_2)(\hat{x}, ns))$ used by Sandier and Serfaty in Chapter 8 of [9] to provide an upper bound on the minimal two-dimensional energy F_ϵ .

A matching lower bound is much more difficult to establish. To obtain it, we prove in Lemma 4.1 that, for a minimizer $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ of $\mathcal{G}_{LD}^{\epsilon, s}$, we have

$$\frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\text{curl} \hat{A}(\hat{x}, x_3) - \text{curl} \hat{A}_n(\hat{x})|^2 d\hat{x} dx_3 \leq o_{\epsilon, s}(1) M_\epsilon, \quad (1.7)$$

where $M_\epsilon = \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}$. The proof of (1.7) uses single layer potential representation formulas for \vec{A} proved by Bauman and Ko in [2] as well as a priori estimates for single layer potentials (see [4] and [10]) and harmonic functions. This estimate plays a crucial role in the proof of the lower bound, as it implies that the three-dimensional integral $\frac{1}{2} \int_D |\text{curl} \hat{A}(\hat{x}, x_3) - h_{ex}|^2 dx$ can be approximated within $o_{\epsilon, s}(1) M_\epsilon$ by the sum of two-dimensional integrals, $\sum_{n=0}^{N-1} \frac{s}{2} \int_{\Omega} |\text{curl} \hat{A}_n(\hat{x}) - h_{ex}|^2 d\hat{x}$. As a result of (1.7),

the energy for a minimizer from the first term in (1.1) plus the magnetic term $\frac{1}{2} \int_D |\hat{\text{curl}} \hat{A} - h_{ex}|^2 dx$ can be approximated to leading order by the sum of $sF_\epsilon(u_n, \hat{A}_n)$. Combining this observation with the lower bound estimate for F_ϵ proved in [9], we obtain the lower bound estimate in Theorem 1.

As a consequence of our proof of Theorem 1, we conclude that the Josephson coupling term (1.5) contributes a lower order energy to the total Lawrence-Doniach energy. In fact, we obtain

Theorem 2. *Assume $|\ln \epsilon| \ll h_{ex} \ll \frac{1}{\epsilon^2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then*

$$\begin{aligned} s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} + \frac{1}{2} \int_{\mathbb{R}^3 \setminus D} |\nabla \times \vec{A} - h_{ex} \vec{e}_3|^2 dx \\ + \frac{1}{2} \int_D \left[\left(\frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3} \right)^2 + \left(\frac{\partial A^1}{\partial x_3} - \frac{\partial A^3}{\partial x_1} \right)^2 \right] dx \leq o_{\epsilon, s}(1) M_\epsilon \end{aligned}$$

as $(\epsilon, s) \rightarrow (0, 0)$.

(See Section 4 for the proof.)

Theorems 1 and 2 and the estimate (1.7) imply a strong influence up to leading order of the two-dimensional energy F_ϵ on the minimal Lawrence-Doniach energy. In fact, we prove in Corollary 4.3 that for a minimizer $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ of the Lawrence-Doniach energy, we have

$$\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) = \left[\sum_{n=0}^{N-1} sF_\epsilon(u_n, \hat{A}_n) \right] [1 + o_{\epsilon, s}(1)].$$

Another consequence of Theorems 1 and 2 is the following:

Corollary. *Under the assumptions of Theorem 1, we have*

$$\left\| \frac{\nabla \times \vec{A}}{h_{ex}} - \vec{e}_3 \right\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} \rightarrow 0,$$

and

$$\frac{1}{N+1} \sum_{n=0}^N \frac{\mu_n}{h_{ex}} \rightarrow d\hat{x} \text{ in } H^{-1}(\Omega)$$

as $(\epsilon, s) \rightarrow (0, 0)$, where μ_n is the vorticity on the n th layer defined as

$$\mu_n = \hat{\text{curl}}(iu_n, \hat{\nabla}_{\hat{A}_n} u_n) + \hat{\text{curl}} \hat{A}_n.$$

(See Corollary 4.4.)

The convergence of the average scaled vorticity in the layers to the Lebesgue measure generalizes a result for minimizers of the two-dimensional Ginzburg-Landau energy F_ϵ studied by Sandier and Serfaty (see Cor. 8.1 in [9]). They showed that for minimizers of F_ϵ , the scaled vorticity measure $\frac{\mu}{h_{ex}}$ converges to $d\hat{x}$ in $H^{-1}(\Omega)$ as $\epsilon \rightarrow 0$. The vorticity measure μ_n in each layer is a gauge-invariant version of the Jacobian determinant of u_n , and is analogous to the vorticity in fluids. If u_n is given in polar coordinates by $\rho_n e^{i\theta_n}$, then $\mu_n = \hat{\text{curl}}\{\rho_n^2(\hat{\nabla}\theta_n - \hat{A}_n)\} + \hat{\text{curl}} \hat{A}_n$. The above corollary indicates that on average there are numerous vortices and they have an approximately uniform distribution. More detailed results on the nature and number of vortices for minimizers of the Lawrence-Doniach energy is an interesting open problem to which the results of this paper should be relevant.

Recall that another model for certain high-temperature anisotropic superconductors is the three-dimensional anisotropic Ginzburg-Landau model. In this model, a mass tensor with unequal principal

values is introduced to account for the anisotropic structure in the superconductor. (See [3] and [7] for more background information.) For a given admissible function (ψ, \vec{A}) in $H^1(D; \mathbb{C}) \times E$, the anisotropic Ginzburg-Landau energy $\mathcal{G}_{AGL}^\epsilon$ is given by

$$\begin{aligned} \mathcal{G}_{AGL}^\epsilon(\psi, \vec{A}) &= \frac{1}{2} \int_D \left[|\hat{\nabla}_{\vec{A}} \psi|^2 + \frac{1}{\lambda^2} \left| \left(\frac{\partial}{\partial x_3} - \imath A^3 \right) \psi \right|^2 \right] dx \\ &\quad + \int_D \frac{(1 - |\psi|^2)^2}{4\epsilon^2} dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \vec{A} - h_{ex} \vec{e}_3|^2 dx. \end{aligned} \quad (1.8)$$

Here λ is the same constant as in the Lawrence-Doniach energy. The connection between the Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon, s}$ and the anisotropic Ginzburg-Landau energy $\mathcal{G}_{AGL}^\epsilon$ when ϵ is fixed and s tends to zero was studied in [2], [3], and [11]. In particular, it was shown in [3] that under this assumption, a subsequence of minimizers of the Lawrence-Doniach energy form a minimizing sequence of the anisotropic Ginzburg-Landau energy. Gamma convergence of the Lawrence-Doniach energy in this case to the anisotropic Ginzburg-Landau energy was proved in [11].

Our last result concerns the asymptotic behavior of the two energies as both ϵ and s tend to zero. We prove that, under an additional assumption on s versus ϵ , the difference between the two minimum energies is negligible compared to the leading term in the minimal Lawrence-Doniach energy. More precisely, we have the following theorem:

Theorem 3. *Assume $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$ and let $(\zeta, \vec{B}) \in H^1(D; \mathbb{C}) \times K$ be a minimizer of $\mathcal{G}_{AGL}^\epsilon$. If in addition we assume that $s \leq C\epsilon$ for all ϵ sufficiently small where C is a constant independent of ϵ , then*

$$|\mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) - \mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A})| \leq o_\epsilon(1) \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}$$

as $\epsilon \rightarrow 0$. Hence

$$|\mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) - \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}| \leq o_\epsilon(1) \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}$$

as $\epsilon \rightarrow 0$.

(See Theorem 5.6 and Theorem 5.7.)

In the case when ϵ is fixed, the discrete nature of the layering in the Lawrence-Doniach model is eliminated as $s \rightarrow 0$ and therefore it is very natural that the discrete model reduces to the continuous one. The situation considered here is more delicate, since the interlayer distance s is allowed to be at the same order as the characteristic vortex size ϵ , in which case the discrete nature of the Lawrence-Doniach model plays a more important role. Note that the assumption $s \leq C\epsilon$ is only needed in Theorem 3. The previous two theorems concerning the minimum Lawrence-Doniach energy hold even for the very discrete case (e.g., $1 \gg s \gg \epsilon$).

Our paper is organized as follows: In Section 2 we state some preliminary results concerning the single layer potential representation formulas for \hat{A} and \hat{A}_n . In Section 3 we prove the upper bound on the minimal Lawrence-Doniach energy. In Section 4 we prove (1.7) and use it to prove the lower bound on the minimal Lawrence-Doniach energy, as well as the corollaries stated above. Finally in Section 5 we prove the comparison result between the minimal Lawrence-Doniach energy and the minimal three-dimensional anisotropic Ginzburg-Landau energy and its consequence as summarized in Theorem 3.

2 Preliminaries

As noted in the introduction, the Lawrence-Doniach energy is invariant under the gauge transformation (1.3) and minimizers of $\mathcal{G}_{LD}^{\epsilon, s}$ are gauge-equivalent to a minimizer in the ‘‘Coulomb gauge’’. It was

proved by Bauman and Ko in [2] that, for a minimizer $(\{u_n\}_{n=0}^N, \vec{A})$ of $\mathcal{G}_{LD}^{\epsilon, s}$ in the “Coulomb gauge”, the magnetic potential \vec{A} has an explicit representation formula using single layer potentials. Recall the definition of the space $\tilde{H}^1(\mathbb{R}^3)$ in the introduction. From [2], each $\vec{C} \in \tilde{H}^1(\mathbb{R}^3)$ has a representative in $L^6(\mathbb{R}^3; \mathbb{R}^3)$ such that

$$\|\vec{C}\|_{L^6(\mathbb{R}^3; \mathbb{R}^3)} \leq 2\|\vec{C}\|_{\tilde{H}^1(\mathbb{R}^3)} \quad (2.1)$$

and

$$\|\vec{C}\|_{\tilde{H}^1(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (|\nabla \cdot \vec{C}|^2 + |\nabla \times \vec{C}|^2) dx. \quad (2.2)$$

We remark that the Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon, s}$ considered here is different from that studied in [3], [2] and [11], via a simple rescaling in the energy and in the magnetic potentials. (The scaling used here is the same as that used by Sandier and Serfaty in [9] which has been very successful in analyzing minimizers for the two-dimensional Ginzburg-Landau energy as ϵ tends to zero.)

More precisely, setting $\kappa = \frac{1}{\epsilon}$ and letting \mathcal{G}_{LD}^κ be the Lawrence-Doniach energy studied in [3] with ψ_n as the order parameter for the n th layer and \vec{A}_{LD} as the magnetic potential for \mathcal{G}_{LD}^κ , respectively, we have

$$\begin{cases} \mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) = \frac{\kappa^2}{2} \mathcal{G}_{LD}^\kappa(\{\psi_n\}_{n=0}^N, \vec{A}_{LD}), \\ u_n = \psi_n \text{ and } \vec{A} = \kappa \vec{A}_{LD}. \end{cases} \quad (2.3)$$

Similar rescaling holds for the anisotropic Ginzburg-Landau energy, i.e.,

$$\begin{cases} \mathcal{G}_{AGL}^\epsilon(\psi, \vec{A}) = \frac{\kappa^2}{2} \mathcal{G}_{EM}^\kappa(\psi_{EM}, \vec{A}_{EM}), \\ \psi = \psi_{EM} \text{ and } \vec{A} = \kappa \vec{A}_{EM}, \end{cases} \quad (2.4)$$

where \mathcal{G}_{EM}^κ is the anisotropic Ginzburg-Landau (or effective mass) energy introduced in [3], and ψ_{EM} and \vec{A}_{EM} are the order parameter and the magnetic potential for \mathcal{G}_{EM}^κ , respectively. The above formulas will be used in Section 5.

The analysis in [2] (after appropriate rescaling) applies here without any difficulty. In particular, we have representation formulas for A^1, A^2, A_n^1 and A_n^2 for a minimizer of the Lawrence-Doniach energy in the Coulomb gauge as in Lemma 3.1, Theorem 3.2 and Corollary 3.3 in [2]. To state these formulas, we first define the single layer potential for our setting. For each $k \in \{0, 1, \dots, N\}$, and for a given function $g \in L^p(\Omega \times \{ks\})$ with $1 < p < \infty$, we define the operator S_k by

$$[S_k(g)](x) = \int_{\Omega \times \{ks\}} \frac{c}{|x - Q|} g(Q) d\sigma(Q)$$

for x in $\mathbb{R}^3 \setminus [\overline{\Omega} \times \{ks\}]$ where $d\sigma$ denotes the surface measure on the plane and $c = -\frac{1}{4\pi}$. (See [4] and [10] for results on layer potentials in smooth and Lipschitz domains, respectively.) Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Define h_k^i in $L^2(\Omega)$ by

$$h_k^i(\hat{x}) = s(\partial_i u_k - \imath A_k^i u_k, -u_k) \chi_\Omega(\hat{x})$$

and define g_k^i in $L^2(\Omega \times \{ks\})$ by

$$g_k^i(x) = \chi_{\Omega \times \{ks\}}(x) h_k^i(\hat{x})$$

for $i = 1, 2$. Then the single layer potential of g_k^i is

$$\begin{aligned} [S_k(g_k^i)](x) &= \int_{\Omega \times \{ks\}} \frac{c}{|x - Q|} g_k^i(Q) d\sigma(Q) \\ &= \int_{\Omega} \frac{c}{|x - (\hat{y}, ks)|} h_k^i(\hat{y}) d\hat{y}. \end{aligned} \quad (2.5)$$

With the above definitions, from the formulas in [2], we have

$$A^i(x) - h_{ex}a^i(x) = \sum_{k=0}^N S_k(g_k^i)(x) \text{ in } L_{loc}^2(\mathbb{R}^3) \quad (2.6)$$

and

$$A_n^i(\hat{x}) - h_{ex}a_n^i(\hat{x}) = t_n^i(\hat{x}, ns) + \sum_{\substack{k=0 \\ k \neq n}}^N S_k(g_k^i)(\hat{x}, ns) \text{ a.e. in } \mathbb{R}^2 \quad (2.7)$$

for $i = 1, 2$, where $t_n^i(\hat{x}, ns)$ is the trace of $[S_n(g_n^i)](x)$ on $\mathbb{R}^2 \times \{ns\}$, which is given by

$$\begin{aligned} t_n^i(\hat{x}, ns) &= \int_{\Omega \times \{ns\}} \frac{c}{|(\hat{x}, ns) - Q|} g_n^i(Q) d\sigma(Q) \\ &= \int_{\Omega} \frac{c}{|\hat{x} - \hat{y}|} h_n^i(\hat{y}) d\hat{y}, \end{aligned}$$

and $a_n^i(\hat{x}) = a^i(\hat{x}, ns)$ corresponds to the trace of a^i in $\mathbb{R}^2 \times \{ns\}$.

In order to state further properties that will be used later, we need some definitions and results from [2] (based on the theory of single layer potentials in [4] and [10]) concerning nontangential limits and nontangential maximal functions. For fixed $R > 0$ and $0 < \theta < \pi/2$, let

$$\Gamma \equiv \Gamma_{R,\theta} = \{x \in \mathbb{R}^3 : |x| < R \text{ and } |x \cdot \vec{e}_3| > |x| \cos \theta\}$$

be a cone nontangential to the plane $\{x_3 = 0\}$ with vertex at the origin. Denote by

$$\Gamma^+ = \{x \in \Gamma : x_3 > 0\} \text{ and } \Gamma^- = \{x \in \Gamma : x_3 < 0\}.$$

For $(\hat{x}, ns) \in \mathbb{R}^2 \times \{ns\}$, let

$$\Gamma(\hat{x}, ns) = \{y \in \mathbb{R}^3 : y - (\hat{x}, ns) \in \Gamma\}.$$

Similarly, denote by

$$\Gamma^+(\hat{x}, ns) = \{y \in \mathbb{R}^3 : y - (\hat{x}, ns) \in \Gamma^+\}$$

and

$$\Gamma^-(\hat{x}, ns) = \{y \in \mathbb{R}^3 : y - (\hat{x}, ns) \in \Gamma^-\}.$$

For a function u defined in $\Gamma(\hat{x}, ns)$, define the nontangential limit (n.t.limit) of $u(y)$ as $y \rightarrow (\hat{x}, ns)$ by

$$\text{n.t.limit } u(y) = \lim_{y \rightarrow (\hat{x}, ns)} \{u(y) : y \in \Gamma(\hat{x}, ns)\},$$

provided the limit exists. Also we have the following definition of the nontangential maximal function of u at (\hat{x}, ns) , denoted by $u^*(\hat{x}, ns) = u_{R,\theta}^*(\hat{x}, ns)$ for each n in $\{0, 1, \dots, N\}$.

$$u^*(\hat{x}, ns) = \sup\{|u(y)| : y \in \Gamma_{R,\theta}(\hat{x}, ns)\}.$$

By Theorem 3.2 in [2], $S_n(g_n^i) \in W_{loc}^{1,2}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3 \setminus \overline{\Omega}_n)$ and $t_n^i(\hat{x}, ns) \in W_{loc}^{1,2}(\mathbb{R}^2 \times \{ns\})$.

Throughout this paper, we let $\theta = \frac{\pi}{4}$ and $R = 1$. Also, let R_0 be a fixed constant satisfying

$$R_0 \geq 2(\text{diam } \Omega),$$

where $\text{diam } \Omega$ is the diameter of Ω . It follows that the nontangential maximal functions of $S_n(g_n^i)$ and $\nabla S_n(g_n^i)$ are in $L_{loc}^2(\mathbb{R}^2 \times \{ns\})$ and

$$\|(S_n(g_n^i))^*\|_{L^2(\Omega \times \{ns\})} + \|(\nabla S_n(g_n^i))^*\|_{L^2(\Omega \times \{ns\})} \leq C \|g_n^i\|_{L^2(\Omega \times \{ns\})} \quad (2.8)$$

for some constant C depending only on R_0 . (This property of the constant C uses the fact that Ω is a subset of a disk of radius R_0 in \mathbb{R}^2 . See [2].) Also $t_n^i(\hat{x}, ns)$ and $\hat{\nabla} t_n^i(\hat{x}, ns)$ are the nontangential limits of $S_n(g_n^i)$ and $\hat{\nabla} S_n(g_n^i)$, respectively, pointwise a.e. in $\mathbb{R}^2 \times \{ns\}$ and in $L_{loc}^2(\mathbb{R}^2 \times \{ns\})$, and we have

$$\nabla S_n(g_n^i)(x) = \int_{\mathbb{R}^2} \frac{-c(x - (\hat{y}, ns))}{|x - (\hat{y}, ns)|^3} h_n^i(\hat{y}) d\hat{y} \text{ a.e. in } \mathbb{R}^3$$

and

$$(\hat{\nabla} t_n^i)(\hat{x}, ns) = P.V. \int_{\mathbb{R}^2} \frac{-c(\hat{x} - \hat{y})}{|\hat{x} - \hat{y}|^3} h_n^i(\hat{y}) d\hat{y} \text{ a.e. in } \mathbb{R}^2 \times \{ns\},$$

where $P.V.$ denotes the principal-valued integral. In addition, we have

$$\|t_n^i\|_{L^2(\Omega)} + \|\hat{\nabla} t_n^i\|_{L^2(\Omega)} \leq C \|h_n^i\|_{L^2(\Omega)} \quad (2.9)$$

for some constant C depending only on R_0 .

The above representation formulas and properties of the single layer potential will be used in Section 4 in our proof of the lower bound for the minimum Lawrence-Diniach energy.

We conclude this section with the following proposition concerning the minimum of the energies GL_ϵ and F_ϵ over bounded simply connected Lipschitz domains, which is a modification of Proposition 3.4 in [9].

Proposition 2.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected Lipschitz domain. Let*

$$X = \{(v, b) \in H^1(\Omega; \mathbb{C}) \times H_{loc}^1(\mathbb{R}^2; \mathbb{R}^2) : (\text{curl} b - h_{ex}) \in L^2(\mathbb{R}^2)\}$$

and

$$X_\Omega = \{(v, b) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}.$$

Then we have

$$\min_{(v,b) \in X} GL_\epsilon(v, b) = \min_{(v,b) \in X_\Omega} F_\epsilon(v, b).$$

Proof. Note that for any function (v, b) in X , we have $(v, b|_\Omega) \in X_\Omega$. From this and the definitions of GL_ϵ and F_ϵ , we obtain

$$\min_{(v,b) \in X} GL_\epsilon(v, b) \geq \min_{(v,b) \in X_\Omega} F_\epsilon(v, b).$$

Given a minimizer $(v, b) \in X_\Omega$ of F_ϵ , let ϕ solve $\Delta \phi = H$, where $H = (\text{curl} b - h_{ex}) \cdot \chi_\Omega \in L^2(\mathbb{R}^2)$. We may take ϕ to be the Newtonian potential of H . By standard estimates for Newtonian potentials, we have $\phi \in H_{loc}^2(\mathbb{R}^2)$. Define $\tilde{b} = (-\partial_2 \phi, \partial_1 \phi) + (0, h_{ex} x_1) \in H_{loc}^1(\mathbb{R}^2; \mathbb{R}^2)$. Direct calculations show that $\text{curl}(\tilde{b}|_\Omega) = \text{curl} b$ in Ω , where $\tilde{b}|_\Omega$ is the restriction of \tilde{b} on Ω . Therefore there exists $f \in H^2(\Omega)$ such that $b|_\Omega = \tilde{b}|_\Omega + \nabla f$ in Ω . Define $\tilde{v} = v e^{if}$. Simple calculations using that $H = \text{curl} \tilde{b} - h_{ex}$ and gauge invariance imply that

$$GL_\epsilon(\tilde{v}, \tilde{b}) = F_\epsilon(\tilde{v}, \tilde{b}|_\Omega) = F_\epsilon(v, b) = \min_{(v,b) \in X_\Omega} F_\epsilon(v, b).$$

Hence $\min_{(v,b) \in X} GL_\epsilon(v, b) \leq \min_{(v,b) \in X_\Omega} F_\epsilon(v, b)$ and equality must hold. \square

3 Upper bound

From here on in the paper, we let C_0 denote any constant that is independent of ϵ , s , Ω , L , D and R_0 for all ϵ and s sufficiently small. Recall that in our notation, \hat{V} denotes a two-dimensional vector $V = (V_1, V_2)$ and \vec{W} denotes a three-dimensional vector $W = (W_1, W_2, W_3)$. Also $\text{curl } \hat{V} = \partial_1 V_2 - \partial_2 V_1$. We will denote by $\hat{\nabla}$ and $\hat{\Delta}$ the operators (∂_1, ∂_2) and $(\partial_1)^2 + (\partial_2)^2$, respectively.

In this section we prove the following upper bound on the minimum Lawrence-Doniach energy:

Theorem 3.1. *Assume $|\ln \epsilon| \ll h_{ex} \ll \frac{1}{\epsilon^2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A})$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then we have*

$$\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) \leq \frac{|D|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} \right) (1 + c(\epsilon, s))$$

for all ϵ and s sufficiently small, where

$$c(\epsilon, s) = \frac{s}{L} + \frac{C}{\ln \frac{1}{\epsilon \sqrt{h_{ex}}}} = o_{\epsilon, s}(1).$$

Here C is a constant depending only on R_0 and L .

The main idea in the proof is to construct a test configuration with vanishing Josephson coupling term, such that its Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon, s}$ from inside the domain D is a sum of $N+1$ identical copies of the two-dimensional Ginzburg-Landau energy F_ϵ in Ω . In this way we may apply the upper bound estimate for the minimal two-dimensional Ginzburg-Landau energy from Proposition 8.1 in [9] to obtain the leading energy of $\mathcal{G}_{LD}^{\epsilon, s}$. The technical difficulty comes from extending the magnetic potential \vec{A} outside the domain D appropriately so that the energy contribution from $\mathbb{R}^3 \setminus D$ is of a lower order compared to that in D .

Proof of Theorem 3.1. We first construct a test configuration of the two-dimensional Ginzburg-Landau energy F_ϵ , as in the proof of Proposition 8.1 in [9]. Let $\theta = \sqrt{\frac{h_{ex}}{2\pi}}$ and $L_\epsilon = \frac{1}{\theta}\mathbb{Z} \times \frac{1}{\theta}\mathbb{Z}$ in \mathbb{R}^2 . Let $h_\epsilon(\hat{x})$ be a solution in \mathbb{R}^2 (periodic with respect to L_ϵ) of

$$-\hat{\Delta} h_\epsilon + h_\epsilon = 2\pi \sum_{b \in L_\epsilon} \delta_b.$$

Define $\rho_\epsilon(\hat{x})$ by

$$\rho_\epsilon(\hat{x}) = \begin{cases} 0 & \text{if } |\hat{x} - b| \leq \epsilon \text{ for some } b \in L_\epsilon, \\ \frac{|\hat{x} - b|}{\epsilon} - 1 & \text{if } \epsilon \leq |\hat{x} - b| \leq 2\epsilon \text{ for some } b \in L_\epsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\hat{A}_\epsilon(\hat{x}) = (A_\epsilon^1(\hat{x}), A_\epsilon^2(\hat{x}))$ be the solution in \mathbb{R}^2 of $\text{curl } \hat{A}_\epsilon = h_\epsilon$, and $\varphi_\epsilon(\hat{x})$ be the solution (well-defined modulo 2π) of $-\hat{\nabla}^\perp h_\epsilon = \hat{\nabla} \varphi_\epsilon - \hat{A}_\epsilon$ in $\mathbb{R}^2 \setminus L_\epsilon$, where $\hat{\nabla}^\perp = (-\partial_2, \partial_1)$. (See the introduction of [8] for a construction of such a φ_ϵ .) Let $u_\epsilon(\hat{x}) = \rho_\epsilon(\hat{x}) e^{i\varphi_\epsilon(\hat{x})}$ in \mathbb{R}^2 . By the proof of Proposition 8.1 in [9], there exists $\hat{x}_0 \in K_\epsilon = (-\frac{1}{2\theta}, \frac{1}{2\theta}) \times (-\frac{1}{2\theta}, \frac{1}{2\theta})$ such that

$$F_\epsilon(u_\epsilon^{\hat{x}_0}, \hat{A}_\epsilon^{\hat{x}_0}) \leq \frac{|\Omega|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} + C_0 \right) \quad (3.1)$$

for all ϵ sufficiently small, where $u_\epsilon^{\hat{x}_0}(\hat{x}) = u_\epsilon(\hat{x} - \hat{x}_0)$ and $\hat{A}_\epsilon^{\hat{x}_0}(\hat{x}) = \hat{A}_\epsilon(\hat{x} - \hat{x}_0)$.

Using the above test configuration $(u_\epsilon^{\hat{x}_0}, \hat{A}_\epsilon^{\hat{x}_0})$ for the two-dimensional Ginzburg-Landau energy F_ϵ , we next construct a test configuration $(\{v_n\}_{n=0}^N, \vec{B})$ in $[H^1(\Omega; \mathbb{C})]^{N+1} \times E$ for the Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon, s}$. Let $\phi = \phi(\hat{x})$ solve

$$\hat{\Delta} \phi(\hat{x}) = H(\hat{x}) \text{ in } \mathbb{R}^2, \quad (3.2)$$

where $H(\hat{x}) = (h_{\epsilon}^{\hat{x}_0}(\hat{x}) - h_{ex}) \cdot \chi_{\Omega}(\hat{x})$ and $h_{\epsilon}^{\hat{x}_0}(\hat{x}) = h_{\epsilon}(\hat{x} - \hat{x}_0)$. Since $h_{\epsilon} \in L_{loc}^p(\mathbb{R}^2)$ for every $p > 0$, we have $H \in L^p(\mathbb{R}^2)$ and is supported in $\bar{\Omega}$. By Theorem 9.9 in [5], we may choose ϕ to be the Newtonian potential of H , i.e., $\phi = \Gamma_2 * H$, where $\Gamma_2(\hat{x}) = \frac{1}{2\pi} \ln |\hat{x}|$ is the fundamental solution of $\hat{\Delta}$ in \mathbb{R}^2 . By standard estimates on the Newtonian potential, we have $\phi \in W_{loc}^{2,p}(\mathbb{R}^2)$ for $1 < p < \infty$ and $\hat{\nabla}\phi = \hat{\nabla}\Gamma_2 * H$. In particular, $\phi \in H_{loc}^2(\mathbb{R}^2)$. Let $R = \max\{2(\text{diam } \Omega), 1\}$ and denote by $B_{\frac{R}{2}}$ a disk in \mathbb{R}^2 , with radius $\frac{R}{2}$ such that $\Omega \subset B_{\frac{R}{2}}$. Let B_R and B_{R+1} be disks concentric with $B_{\frac{R}{2}}$ with radii R and $R+1$ respectively. To construct \vec{B} , we need two cut-off functions. Choose $\xi = \xi(\hat{x}) \in C_0^{\infty}(\mathbb{R}^2)$ such that $\xi \geq 0$, $\xi(\hat{x}) \equiv 1$ for $\hat{x} \in B_R$ and $\xi(\hat{x}) \equiv 0$ for $\hat{x} \in \mathbb{R}^2 \setminus B_{R+1}$ and $|\hat{\nabla}\xi| \leq 2$. Let $\eta = \eta(x_3) \in C_0^{\infty}(\mathbb{R})$ be such that $\eta \geq 0$, η is symmetric with respect to $x_3 = \frac{L}{2}$, $\eta(x_3) \equiv 1$ if $-\frac{s}{2} \leq x_3 \leq L + \frac{s}{2}$ and $\eta(x_3) \equiv 0$ if $x_3 \leq -\frac{s}{2} - d$ or $x_3 \geq L + \frac{s}{2} + d$ satisfying $|\eta'(x_3)| \leq \frac{2}{d}$ for some fixed positive number d . Now define

$$\vec{B}(x) = h_{ex}\vec{a}(x) + \eta(x_3)\xi(\hat{x})(-\partial_2\phi(\hat{x}), \partial_1\phi(\hat{x}), 0) \in H_{loc}^1(\mathbb{R}^3; \mathbb{R}^3),$$

where recall that $\vec{a}(x) = (0, x_1, 0)$ satisfies $\nabla \times \vec{a} = \vec{e}_3$ in \mathbb{R}^3 . By simple calculations we have

$$\nabla \times \vec{B} = h_{ex}\vec{e}_3 + (-\eta'\xi\partial_1\phi, -\eta'\xi\partial_2\phi, \eta(\xi\hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)). \quad (3.3)$$

Note that for $x = (\hat{x}, x_3) \in \Omega \times [0, L]$, $\vec{B}(x) = [h_{ex}\vec{a} + (-\partial_2\phi, \partial_1\phi, 0)](x)$ is independent of x_3 . Denote by \hat{B}_{Ω} the restriction of $\hat{B} = (B^1, B^2)$ on $\Omega \times \{x_3\}$ for any $x_3 \in [0, L]$, i.e.,

$$\hat{B}_{\Omega} = (-\partial_2\phi \cdot \chi_{\Omega}(\hat{x}), (h_{ex}x_1 + \partial_1\phi) \cdot \chi_{\Omega}(\hat{x})).$$

Clearly \hat{B}_{Ω} satisfies

$$\text{curl}\hat{B}_{\Omega} = h_{ex} + (h_{\epsilon}^{\hat{x}_0} - h_{ex}) = h_{\epsilon}^{\hat{x}_0} = \text{curl}\hat{A}_{\epsilon}^{\hat{x}_0}$$

on Ω . Since Ω is simply connected, there exists a function $f \in H^2(\Omega)$ such that $\hat{B}_{\Omega} = \hat{A}_{\epsilon}^{\hat{x}_0} + \hat{\nabla}f$. Define $v_n(\hat{x}) = v(\hat{x}) \equiv u_{\epsilon}^{\hat{x}_0}(\hat{x})e^{if}$ for all $n = 0, 1, \dots, N$. Since (v, \hat{B}_{Ω}) is gauge equivalent to $(u_{\epsilon}^{\hat{x}_0}, \hat{A}_{\epsilon}^{\hat{x}_0})$ for the two-dimensional Ginzburg-Landau energy F_{ϵ} , by (3.1), we have

$$F_{\epsilon}(v, \hat{B}_{\Omega}) = F_{\epsilon}(u_{\epsilon}^{\hat{x}_0}, \hat{A}_{\epsilon}^{\hat{x}_0}) \leq \frac{|\Omega|}{2} h_{ex} (\ln \frac{1}{\epsilon\sqrt{h_{ex}}} + C_0). \quad (3.4)$$

We show that the test configuration $(\{v_n\}_{n=0}^N, \vec{B})$ defined above gives us the desired upper bound for the Lawrence-Doniach energy $\mathcal{G}_{LD}^{\epsilon, s}$.

It is easy to see that, since the v_n 's are all equal and $B^3 \equiv 0$, the Josephson coupling term (1.5) vanishes. Therefore

$$\begin{aligned} \mathcal{G}_{LD}^{\epsilon, s}(\{v_n\}_{n=0}^N, \vec{B}) &= s \sum_{n=0}^N \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}_n} v_n|^2 + \frac{1}{4\epsilon^2} (1 - |v_n|^2)^2 \right] d\hat{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \vec{B} - h_{ex}\vec{e}_3|^2 dx. \end{aligned}$$

It follows from (3.3) and the definitions of ξ and η that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \vec{B} - h_{ex}\vec{e}_3|^2 dx &= \frac{1}{2} \int_{\mathbb{R}^3} \left[|\eta'|^2 |\xi|^2 |\hat{\nabla}\phi|^2 + \eta^2 (\xi\hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)^2 \right] dx \\ &= \frac{1}{2} \int_{-\frac{s}{2}-d}^{-\frac{s}{2}} \int_{\mathbb{R}^2} \left[|\eta'|^2 |\xi|^2 |\hat{\nabla}\phi|^2 + \eta^2 (\xi\hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)^2 \right] d\hat{x} dx_3 \\ &\quad + \frac{1}{2} \int_{L+\frac{s}{2}}^{L+\frac{s}{2}+d} \int_{\mathbb{R}^2} \left[|\eta'|^2 |\xi|^2 |\hat{\nabla}\phi|^2 + \eta^2 (\xi\hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)^2 \right] d\hat{x} dx_3 \\ &\quad + \frac{1}{2} \int_{-\frac{s}{2}}^{L+\frac{s}{2}} \int_{\mathbb{R}^2} (\xi\hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)^2 d\hat{x} dx_3. \end{aligned}$$

Since $(\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2$ does not depend on x_3 and $Ns = L$, we have

$$\frac{1}{2} \int_{-\frac{s}{2}}^{L+\frac{s}{2}} \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} dx_3 = s(N+1) \cdot \frac{1}{2} \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x}.$$

Also since $\hat{B}_n = \hat{B}_\Omega$ and $v_n = v$ for all $n = 0, 1, \dots, N$, we have

$$\begin{aligned} & s \sum_{n=0}^N \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}_n} v_n|^2 + \frac{1}{4\epsilon^2} (1 - |v_n|^2)^2 \right] d\hat{x} \\ &= s(N+1) \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}_\Omega} v|^2 + \frac{1}{4\epsilon^2} (1 - |v|^2)^2 \right] d\hat{x}. \end{aligned}$$

Therefore we may write $\mathcal{G}_{LD}^{\epsilon, s}(\{v_n\}_{n=0}^N, \vec{B})$ as a sum:

$$\mathcal{G}_{LD}^{\epsilon, s}(\{v_n\}_{n=0}^N, \vec{B}) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= s(N+1) \left\{ \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}_\Omega} v|^2 + \frac{1}{4\epsilon^2} (1 - |v|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \right\}, \\ I_2 &= \frac{1}{2} \int_{L+\frac{s}{2}}^{L+\frac{s}{2}+d} \int_{\mathbb{R}^2} \left[|\eta'|^2 |\xi|^2 |\hat{\nabla} \phi|^2 + \eta^2 (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 \right] d\hat{x} dx_3, \end{aligned}$$

and

$$I_3 = \frac{1}{2} \int_{-\frac{s}{2}-d}^{-\frac{s}{2}} \int_{\mathbb{R}^2} \left[|\eta'|^2 |\xi|^2 |\hat{\nabla} \phi|^2 + \eta^2 (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 \right] d\hat{x} dx_3.$$

By the symmetry property of η , it is obvious that $I_2 = I_3$. In the following lemmas, we prove several estimates from which we obtain (in Lemma 3.5) that

$$I_1 + 2I_2 \leq \frac{|D|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} \right) \left(1 + \frac{s}{L} + \frac{C}{\ln \frac{1}{\epsilon \sqrt{h_{ex}}}} \right)$$

for all ϵ and s sufficiently small where C is a constant depending only on R_0 and L . This concludes the proof. \square

Our first lemma concerns the L^2 norm of the function $H(\hat{x}) = (h_{\epsilon}^{\hat{x}_0}(\hat{x}) - h_{ex}) \cdot \chi_{\Omega}(\hat{x})$. Note that H is independent of s .

Lemma 3.2. *For the function H defined above, we have*

$$\|H\|_{L^2(\mathbb{R}^2)}^2 = \|h_{\epsilon}^{\hat{x}_0} - h_{ex}\|_{L^2(\Omega)}^2 \leq C_0 |\Omega| h_{ex}$$

for all ϵ sufficiently small and C_0 as described above.

Proof. Let $K_{\epsilon} = (-\frac{1}{2\theta}, \frac{1}{2\theta}) \times (-\frac{1}{2\theta}, \frac{1}{2\theta})$ and $K_{\epsilon}^{\hat{x}_0}$ be the translation of K_{ϵ} by \hat{x}_0 . Then we show that for all ϵ sufficiently small,

$$\|H\|_{L^2(\mathbb{R}^2)}^2 \leq 2 \frac{|\Omega|}{|K_{\epsilon}^{\hat{x}_0}|} \|h_{\epsilon}^{\hat{x}_0} - h_{ex}\|_{L^2(K_{\epsilon}^{\hat{x}_0})}^2 = 2 \frac{|\Omega|}{|K_{\epsilon}|} \|h_{\epsilon} - h_{ex}\|_{L^2(K_{\epsilon})}^2. \quad (3.5)$$

Indeed, let $\{K_i\}$ be the collection of cubes formed by the lattice L_ϵ in \mathbb{R}^2 such that $K_i \cap \Omega \neq \emptyset$ and $\Omega \subset \bigcup_i K_i$. Since $h_\epsilon^{\hat{x}_0} - h_{ex}$ is periodic with respect to L_ϵ , we have

$$\|H\|_{L^2(\mathbb{R}^2)}^2 = \|h_\epsilon^{\hat{x}_0} - h_{ex}\|_{L^2(\Omega)}^2 \leq \sum_i \|h_\epsilon^{\hat{x}_0} - h_{ex}\|_{L^2(K_i)}^2 = \frac{\sum_i |K_i|}{|K_\epsilon^{\hat{x}_0}|} \|h_\epsilon^{\hat{x}_0} - h_{ex}\|_{L^2(K_\epsilon^{\hat{x}_0})}^2.$$

When ϵ is sufficiently small, $\sum_i |K_i| \leq 2|\Omega|$, from which (3.5) follows immediately. By Proposition 3.2 in [8], we have

$$\|h_\epsilon - h_{ex}\|_{L^2(K_\epsilon)}^2 \leq C_0.$$

Combining the above inequality with claim (3.5) and using the fact that $|K_\epsilon| = (\frac{1}{\theta})^2 = \frac{2\pi}{h_{ex}}$, we have

$$\|H\|_{L^2(\mathbb{R}^2)}^2 \leq 2C_0 \frac{|\Omega|}{|K_\epsilon|} = 2C_0 \frac{h_{ex}}{2\pi} |\Omega| = C_0 |\Omega| h_{ex}$$

for all ϵ sufficiently small. \square

Lemma 3.3. *Let $\phi(\hat{x})$ be the Newtonian potential of H in \mathbb{R}^2 and $\xi(\hat{x})$ be defined as above. Then we have*

$$\int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq C |\Omega| h_{ex}$$

for all ϵ sufficiently small and some constant C depending only on R_0 .

Proof. Recall that $\hat{\nabla} \phi(\hat{x}) = \int_{\Omega} \hat{\nabla}_{\hat{x}} \Gamma_2(\hat{x} - \hat{y}) H(\hat{y}) d\hat{y}$ and ξ and ϕ are independent of s . Also recall that, by the definition of ξ , $\hat{\nabla} \xi$ is supported on $\overline{B_{R+1}} \setminus B_R$ and $|\hat{\nabla} \xi|^2 \leq 4$. Therefore for all ϵ sufficiently small,

$$\int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} = \int_{B_{R+1} \setminus B_R} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq 4 \int_{B_{R+1} \setminus B_R} \left| \int_{\Omega} \hat{\nabla}_{\hat{x}} \Gamma_2(\hat{x} - \hat{y}) H(\hat{y}) d\hat{y} \right|^2 d\hat{x}.$$

It follows from the inequality $|\hat{\nabla}_{\hat{x}} \Gamma_2(\hat{x} - \hat{y})| \leq \frac{C_0}{|\hat{x} - \hat{y}|}$ that

$$\int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq C_0 \int_{B_{R+1} \setminus B_R} \left[\int_{\Omega} \frac{1}{|\hat{x} - \hat{y}|} \cdot |H(\hat{y})| d\hat{y} \right]^2 d\hat{x}.$$

Since $\Omega \subset B_{\frac{R}{2}}$, we have $|\hat{x} - \hat{y}| \geq \frac{R}{2}$, which along with Hölder's inequality implies

$$\left[\int_{\Omega} \frac{1}{|\hat{x} - \hat{y}|} \cdot |H(\hat{y})| d\hat{y} \right]^2 \leq \frac{4}{R^2} |\Omega| \cdot \|H\|_{L^2(\Omega)}^2$$

and thus

$$\int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq C_0 \cdot |B_{R+1} \setminus B_R| \cdot \frac{4}{R^2} |\Omega| \cdot \|H\|_{L^2(\Omega)}^2 = C_0 \cdot \frac{(2R+1)}{R^2} \cdot |\Omega| \cdot \|H\|_{L^2(\Omega)}^2.$$

Since $R \geq 1$, it is clear that $\frac{2R+1}{R^2} = \frac{2}{R} + \frac{1}{R^2} \leq 3$. Hence, it follows from Lemma 3.2 that

$$\int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq C_0 |\Omega| \cdot \|H\|_{L^2(\Omega)}^2 \leq C |\Omega| h_{ex}$$

for all ϵ sufficiently small and some constant C depending only on R_0 . \square

Lemma 3.4. *For I_1 as defined above, we have*

$$I_1 \leq \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + \frac{s}{L} \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + C \frac{|D|}{2} h_{ex} \quad (3.6)$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 .

Proof. Using the equation (3.2) and the definition of $\xi(\hat{x})$, we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} &\leq \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi)^2 d\hat{x} + \int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \\ &= \int_{\Omega} (h_{\epsilon}^{\hat{x}_0} - h_{ex})^2 d\hat{x} + \int_{\mathbb{R}^2} (\hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x}. \end{aligned}$$

From this and lemmas 3.2 and 3.3 we obtain

$$\frac{1}{2} \int_{\mathbb{R}^2} (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} \leq C |\Omega| h_{ex}. \quad (3.7)$$

Thus from the definition of I_1 ,

$$\begin{aligned} I_1 &\leq s(N+1) \left\{ \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}_{\Omega}} v|^2 + \frac{1}{4\epsilon^2} (1 - |v|^2)^2 \right] d\hat{x} + C |\Omega| h_{ex} \right\} \\ &\leq s(N+1) F_{\epsilon}(v, \hat{B}_{\Omega}) + s(N+1) \cdot C |\Omega| h_{ex}. \end{aligned}$$

By (3.4) and the identity $sN|\Omega| = |D|$ we have

$$\begin{aligned} I_1 &\leq s(N+1) \frac{|\Omega|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} + C_0 \right) + s(N+1) \cdot C |\Omega| h_{ex} \\ &\leq \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + \frac{s}{L} \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + C \frac{|D|}{2} h_{ex} \end{aligned}$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 . \square

Lemma 3.5. *I_2 defined as above satisfies $I_2 \leq C \frac{|D|}{2} h_{ex}$ and*

$$I_1 + 2I_2 \leq \frac{|D|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} \right) \left(1 + \frac{s}{L} + \frac{C}{\ln \frac{1}{\epsilon \sqrt{h_{ex}}}} \right)$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 and L .

Proof. We write I_2 as

$$I_2 = I_{2,1} + I_{2,2},$$

where

$$I_{2,1} = \frac{1}{2} \int_{L+\frac{s}{2}}^{L+\frac{s}{2}+d} \int_{\mathbb{R}^2} |\eta'|^2 |\xi|^2 |\hat{\nabla} \phi|^2 d\hat{x} dx_3$$

and

$$I_{2,2} = \frac{1}{2} \int_{L+\frac{s}{2}}^{L+\frac{s}{2}+d} \int_{\mathbb{R}^2} \eta^2 (\xi \hat{\Delta} \phi + \hat{\nabla} \xi \cdot \hat{\nabla} \phi)^2 d\hat{x} dx_3.$$

Recall that, by our choice of η and ξ , we have $|\eta'| \leq \frac{2}{d}$ and $|\xi| \leq 1$ is supported on $\overline{B_{R+1}}$. Therefore

$$I_{2,1} \leq \frac{1}{2} \frac{4}{d^2} d \int_{\mathbb{R}^2} |\xi|^2 |\hat{\nabla} \phi|^2 d\hat{x} \leq \frac{2}{d} \int_{B_{R+1}} |\hat{\nabla} \phi|^2 d\hat{x}.$$

Using $\hat{\nabla}\phi(\hat{x}) = \int_{\Omega} \hat{\nabla}_{\hat{x}}\Gamma_2(\hat{x} - \hat{y})H(\hat{y})d\hat{y}$ and Hölder's inequality, we obtain

$$\int_{B_{R+1}} |\hat{\nabla}\phi|^2 d\hat{x} \leq \int_{B_{R+1}} \left(\int_{\Omega} |\hat{\nabla}_{\hat{x}}\Gamma_2(\hat{x} - \hat{y})| d\hat{y} \right) \left(\int_{\Omega} |\hat{\nabla}_{\hat{x}}\Gamma_2(\hat{x} - \hat{y})| \cdot |H(\hat{y})|^2 d\hat{y} \right) d\hat{x}.$$

For any fixed $\hat{x} \in B_{R+1}$ and for every $\hat{y} \in \Omega \subset B_{\frac{R}{2}}$, we have $0 \leq |\hat{x} - \hat{y}| \leq (R+1) + \frac{R}{2} = \frac{3R}{2} + 1 \leq \frac{5R}{2}$ since $R \geq 1$. Letting $\hat{z} = \hat{x} - \hat{y}$, we get

$$\int_{\Omega} |\hat{\nabla}_{\hat{x}}\Gamma_2(\hat{x} - \hat{y})| d\hat{y} \leq C_0 \int_{\Omega} \frac{1}{|\hat{x} - \hat{y}|} d\hat{y} \leq C_0 \int_{B(0, \frac{5R}{2})} \frac{1}{|\hat{z}|} d\hat{z} = C_0 \cdot R, \quad (3.8)$$

where $B(0, \frac{5R}{2})$ is the disk centered at the origin with radius $\frac{5R}{2}$ in \mathbb{R}^2 . Similarly, by the symmetry of Γ_2 in \hat{x} and \hat{y} and almost exactly the same arguments, we get

$$\int_{B_{R+1}} |\hat{\nabla}_{\hat{x}}\Gamma_2(\hat{x} - \hat{y})| d\hat{x} \leq C_0 \cdot R \quad (3.9)$$

for any $\hat{y} \in \Omega$. It follows from (3.8), Fubini's theorem, (3.9) and Lemma 3.2 that

$$I_{2,1} \leq \frac{2}{d} (C_0 R)^2 \cdot \|H\|_{L^2(\Omega)}^2 \leq \frac{C_0 R^2}{d} \cdot \frac{|\Omega|}{2} h_{ex}.$$

For $I_{2,2}$ we use (3.7) and the fact that $|\eta| \leq 1$ to get

$$I_{2,2} \leq \frac{d}{2} \int_{\mathbb{R}^2} (\xi \hat{\Delta}\phi + \hat{\nabla}\xi \cdot \hat{\nabla}\phi)^2 d\hat{x} \leq d \cdot C \frac{|\Omega|}{2} h_{ex}.$$

Therefore we obtain

$$I_2 = I_{2,1} + I_{2,2} \leq C \left(\frac{R^2}{d} + d \right) \frac{|\Omega|}{2} h_{ex} \leq C \frac{|D|}{2} h_{ex} \quad (3.10)$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 and L .

Now combining inequalities (3.6) and (3.10) we have

$$\begin{aligned} I_1 + 2I_2 &\leq \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + \frac{s}{L} \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} + C \frac{|D|}{2} h_{ex} \\ &= \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 + c(\epsilon, s)), \end{aligned}$$

where

$$c(\epsilon, s) = \frac{s}{L} + \frac{C}{\ln \frac{1}{\epsilon \sqrt{h_{ex}}}}$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 and L . \square

4 Lower bound

In this section we prove the lower bound in Theorem 1. This relies on approximating the energy of the magnetic term $|\text{curl}\hat{A} - h_{ex}|^2$ by the sum of its traces $|\text{curl}\hat{A}_n - h_{ex}|^2$ on the layers Ω_n in the thin domains $\Omega \times [ns, (n+1)s)$. We first show that the error from this approximation is indeed of a lower order compared to the leading order term of the total energy.

Lemma 4.1. Assume $|\ln \epsilon| \ll h_{ex} \ll \frac{1}{\epsilon^2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then we have

$$\frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x})|^2 d\hat{x} dx_3 \leq C s^{\frac{2}{7}} M_{\epsilon}$$

for all ϵ and s sufficiently small, where $M_{\epsilon} = \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}$ and C is a constant depending only on R_0 and L .

Proof. We shall use the single layer potential representation formulas for $\vec{A}(x) - h_{ex} \vec{a}(x)$ and $\hat{A}_n(\hat{x}) - h_{ex} \hat{a}_n(\hat{x})$ proved by Bauman and Ko in [2] for $(\{u_n\}_{n=0}^N, \vec{A})$ as above. First note that since $\vec{a}(x) = (0, x_1, 0)$ is independent of x_3 , we have

$$\hat{A}(\hat{x}, x_3) - \hat{A}_n(\hat{x}) = \left(\hat{A}(\hat{x}, x_3) - h_{ex} \hat{a}(\hat{x}, x_3) \right) - \left(\hat{A}_n(\hat{x}) - h_{ex} \hat{a}_n(\hat{x}) \right)$$

for (\hat{x}, x_3) in $\Omega \times [ns, (n+1)s)$. Therefore

$$\begin{aligned} \hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x}) &= \hat{\text{curl}} (\hat{A} - h_{ex} \hat{a})(\hat{x}, x_3) - \hat{\text{curl}} (\hat{A}_n - h_{ex} \hat{a}_n)(\hat{x}) \\ &= \frac{\partial}{\partial x_1} ((A^2 - h_{ex} a^2)(\hat{x}, x_3) - (A_n^2 - h_{ex} a_n^2)(\hat{x})) \\ &\quad - \frac{\partial}{\partial x_2} ((A^1 - h_{ex} a^1)(\hat{x}, x_3) - (A_n^1 - h_{ex} a_n^1)(\hat{x})) \end{aligned}$$

and it follows from this, (2.6) and (2.7) and the regularity results described in Section 2 that

$$\begin{aligned} &\hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x}) \\ &= \left\{ \sum_{\substack{k=0 \\ k \neq n}}^N \frac{\partial}{\partial x_1} (S_k(g_k^2)(\hat{x}, x_3) - S_k(g_k^2)(\hat{x}, ns)) + \frac{\partial}{\partial x_1} (S_n(g_n^2)(\hat{x}, x_3) - t_n^2(\hat{x}, ns)) \right\} \\ &\quad - \left\{ \sum_{\substack{k=0 \\ k \neq n}}^N \frac{\partial}{\partial x_2} (S_k(g_k^1)(\hat{x}, x_3) - S_k(g_k^1)(\hat{x}, ns)) + \frac{\partial}{\partial x_2} (S_n(g_n^1)(\hat{x}, x_3) - t_n^1(\hat{x}, ns)) \right\} \end{aligned}$$

in $L_{loc}^2(\mathbb{R}^3)$, where $h_k^i(\hat{x}) = s(\partial_i u_k - \imath A_k^i u_k, -\imath u_k) \chi_{\Omega}(\hat{x})$ and $g_k^i(x) = \chi_{\Omega \times \{ks\}}(x) h_k^i(\hat{x})$ for $i = 1, 2$. Since $|u_k| \leq 1$ a.e. in Ω (see [2]), we have

$$|h_k^i| \leq s |\partial_i u_k - \imath A_k^i u_k|$$

and

$$\|g_k^i\|_{L^2(\Omega \times \{ks\})}^2 = \|h_k^i\|_{L^2(\Omega)}^2 \leq s^2 \|\partial_i u_k - \imath A_k^i u_k\|_{L^2(\Omega)}^2.$$

Applying the elementary inequality $(a-b)^2 \leq 2a^2 + 2b^2$ to the representation formula for $\hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x})$ and taking the sum of the integrals, we obtain

$$\frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x})|^2 d\hat{x} dx_3 \leq E_1 + E_2,$$

where

$$E_1 = \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} \left| \sum_{\substack{k=0 \\ k \neq n}}^N \frac{\partial}{\partial x_1} (S_k(g_k^2)(\hat{x}, x_3) - S_k(g_k^2)(\hat{x}, ns)) \right. \\ \left. + \frac{\partial}{\partial x_1} (S_n(g_n^2)(\hat{x}, x_3) - t_n^2(\hat{x}, ns)) \right|^2 d\hat{x} dx_3$$

and

$$E_2 = \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} \left| \sum_{\substack{k=0 \\ k \neq n}}^N \frac{\partial}{\partial x_2} (S_k(g_k^1)(\hat{x}, x_3) - S_k(g_k^1)(\hat{x}, ns)) \right. \\ \left. + \frac{\partial}{\partial x_2} (S_n(g_n^1)(\hat{x}, x_3) - t_n^1(\hat{x}, ns)) \right|^2 d\hat{x} dx_3.$$

In the following we analyze E_1 and the analysis for E_2 will be similar. First define

$$\Delta_{n,k}(\hat{x}, x_3) = \frac{\partial}{\partial x_1} [S_k(g_k^2)(\hat{x}, x_3) - S_k(g_k^2)(\hat{x}, ns)]$$

for $n \neq k$ and

$$\Delta_{n,n}(\hat{x}, x_3) = \frac{\partial}{\partial x_1} [S_n(g_n^2)(\hat{x}, x_3) - t_n^2(\hat{x}, ns)].$$

Note that for $n \neq k$, $\Delta_{n,k}$ is C^∞ in $\mathbb{R}^3 \setminus \overline{\Omega}_k$ since it is harmonic there. By the Cauchy-Schwartz inequality,

$$E_1 \leq (N+1) \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} \sum_{k=0}^N |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3 \\ = (N+1) \sum_{k=0}^N \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3. \quad (4.1)$$

Let $\frac{1}{2} < \alpha < 1$ be a constant to be chosen later. For every k fixed in $\{0, 1, \dots, N\}$, we write

$$\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3 = E_{1,1} + E_{1,2}, \quad (4.2)$$

where

$$E_{1,1} = E_{1,1,k} = \sum_{|n-k| \leq s^{-\alpha}} \int_{ns}^{(n+1)s} \int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3$$

and

$$E_{1,2} = E_{1,2,k} = \sum_{|n-k| > s^{-\alpha}} \int_{ns}^{(n+1)s} \int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3.$$

The sums above are taken over n in the indicated subsets of $\{0, 1, \dots, N-1\}$. If $|n-k| \leq s^{-\alpha}$, we have $|ns - ks| \leq s^{1-\alpha} \rightarrow 0$ as $s \rightarrow 0$. Therefore, for s sufficiently small and for each $n \neq k$ in $\{0, 1, \dots, N-1\}$ satisfying $|n-k| \leq s^{-\alpha}$, the following holds for any $(\hat{x}, x_3) \in \Omega \times [ns, (n+1)s)$:

$$|\Delta_{n,k}(\hat{x}, x_3)| \leq \left| \frac{\partial}{\partial x_1} [S_k(g_k^2)(\hat{x}, x_3)] \right| + \left| \frac{\partial}{\partial x_1} [S_k(g_k^2)(\hat{x}, ns)] \right| \\ \leq 2 \left[\frac{\partial}{\partial x_1} S_k(g_k^2) \right]^*(\hat{x}, ks)$$

and thus

$$\int_{ns}^{(n+1)s} \int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} dx_3 \leq 4s \left\| \left[\frac{\partial}{\partial x_1} S_k(g_k^2) \right]^*(\hat{x}, ks) \right\|_{L^2(\Omega \times \{ks\})}^2,$$

where from Section 2, $\left[\frac{\partial}{\partial x_1} S_k(g_k^2) \right]^*(\hat{x}, ks)$ is the nontangential maximal function of the tangential derivative $\frac{\partial}{\partial x_1} S_k(g_k^2)$ at the point (\hat{x}, ks) on the k th layer Ω_k . By (2.8) we have

$$\left\| \left[\frac{\partial}{\partial x_1} S_k(g_k^2) \right]^* \right\|_{L^2(\Omega \times \{ks\})}^2 \leq C \|g_k^2\|_{L^2(\Omega \times \{ks\})}^2 \leq Cs^2 \|\partial_2 u_k - \imath A_k^2 u_k\|_{L^2(\Omega)}^2,$$

where C is a constant depending only on R_0 . For $n = k$, we know from (2.9) that

$$\|\hat{\nabla} t_k^2\|_{L^2(\Omega)} \leq C \|g_k^2\|_{L^2(\Omega \times \{ks\})}.$$

Hence

$$\begin{aligned} E_{1,1} &\leq \sum_{|n-k| \leq s^{-\alpha}} 4sC \|g_k^2\|_{L^2(\Omega \times \{ks\})}^2 \leq C \cdot s^{-\alpha} \cdot s \|g_k^2\|_{L^2(\Omega \times \{ks\})}^2 \\ &= C \cdot s^{1-\alpha} \|g_k^2\|_{L^2(\Omega \times \{ks\})}^2 \end{aligned} \quad (4.3)$$

for all s sufficiently small and some constant C depending only on R_0 .

In order to estimate $E_{1,2}$, consider n in $\{0, 1, \dots, N-1\}$ such that $|n-k| > s^{-\alpha}$. Recall that $S_k(g_k^i)$ is harmonic in $\mathbb{R}^3 \setminus \overline{\Omega}_k$. Without loss of generality, we may assume $k + s^{-\alpha} < n \leq N-1$. (The analysis for $0 \leq n < k - s^{-\alpha}$ is similar.) Let $D_U = \{(\hat{x}, x_3) \in D : x_3 \geq ks + s^{1-\alpha}\}$. Then it is clear that $\Omega \times [ns, (n+1)s] \subset D_U$ for every n satisfying the above assumptions. Take some bounded smooth domain $D_k \subset \{(\hat{x}, x_3) \in \mathbb{R}^3 : x_3 > ks + \frac{s^{1-\alpha}}{2}\}$ in \mathbb{R}^3 such that $\overline{\Omega} \times \{ks + \frac{s^{1-\alpha}}{2}\}$ is a flat portion of the boundary of D_k , $\overline{D_U} \subset D_k$ and $\text{dist}(\overline{D_U}, \partial D_k) \geq \frac{s^{1-\alpha}}{2}$. Then $S_k(g_k^i)$ is harmonic in D_k and for each $x \in D_k$, it follows from (2.5) and Hölder's inequality that

$$\begin{aligned} |S_k(g_k^i)(x)| &= \left| \int_{\Omega} \frac{c}{|x - (\hat{y}, ks)|} h_k^i(\hat{y}) d\hat{y} \right| \\ &\leq \frac{1}{4\pi} \int_{\Omega} \frac{1}{|x_3 - ks|} \cdot |h_k^i(\hat{y})| d\hat{y} \\ &\leq \frac{1}{4\pi} \int_{\Omega} \frac{2}{s^{1-\alpha}} \cdot |h_k^i(\hat{y})| d\hat{y} \leq \frac{C_0}{s^{1-\alpha}} |\Omega|^{\frac{1}{2}} \|h_k^i\|_{L^2(\Omega)}, \end{aligned}$$

and therefore $\sup_{D_k} |S_k(g_k^i)| \leq \frac{C_0}{s^{1-\alpha}} |\Omega|^{\frac{1}{2}} \|h_k^i\|_{L^2(\Omega)}$. For $(\hat{x}, x_3) \in \overline{\Omega} \times [ns, (n+1)s]$, we have (since $(\hat{x}, x_3) \in \overline{\Omega} \times [ns, (n+1)s] \subset \subset \mathbb{R}^3 \setminus \overline{\Omega}_k$ and $\Delta_{n,k}$ is harmonic in $\mathbb{R}^3 \setminus \overline{\Omega}_k$)

$$\begin{aligned} |\Delta_{n,k}(\hat{x}, x_3)| &\leq \sup_{\overline{\Omega} \times [ns, (n+1)s]} \left| \frac{\partial^2}{\partial x_1 \partial x_3} S_k(g_k^2) \right| \cdot |x_3 - ns| \\ &\leq s \cdot \sup_{D_U} \left| \frac{\partial^2}{\partial x_1 \partial x_3} S_k(g_k^2) \right|. \end{aligned}$$

By Theorem 2.10 in [5] and the fact that $\text{dist}(\overline{D_U}, \partial D_k) \geq \frac{s^{1-\alpha}}{2}$, we have

$$\sup_{D_U} \left| \frac{\partial^2}{\partial x_1 \partial x_3} S_k(g_k^2) \right| \leq \left(\frac{12}{s^{1-\alpha}} \right)^2 \sup_{D_k} |S_k(g_k^2)| \leq \frac{C_0}{s^{3-3\alpha}} |\Omega|^{\frac{1}{2}} \|h_k^2\|_{L^2(\Omega)}.$$

Hence we obtain

$$|\Delta_{n,k}(\hat{x}, x_3)| \leq C_0 s^{3\alpha-2} |\Omega|^{\frac{1}{2}} \|h_k^2\|_{L^2(\Omega)}$$

and therefore

$$\int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} \leq C_0 s^{6\alpha-4} |\Omega| \cdot \|h_k^2\|_{L^2(\Omega)}^2 \cdot |\Omega| = C_0 |\Omega|^2 s^{6\alpha-4} \|h_k^2\|_{L^2(\Omega)}^2.$$

To get the best rate of convergence in s as $s \rightarrow 0$, we may take $\alpha = \frac{5}{7}$ so that $1 - \alpha = 6\alpha - 4 = \frac{2}{7}$. The above estimate then becomes

$$\int_{\Omega} |\Delta_{n,k}(\hat{x}, x_3)|^2 d\hat{x} \leq C_0 |\Omega|^2 s^{\frac{2}{7}} \|h_k^2\|_{L^2(\Omega)}^2.$$

Integrating over $[ns, (n+1)s]$ and observing that the cardinality of the indices for the summation on n in $E_{1,2}$ is less than N and $Ns = L$ is fixed, we obtain

$$E_{1,2} \leq L \cdot C_0 |\Omega|^2 s^{\frac{2}{7}} \|h_k^2\|_{L^2(\Omega)}^2 \leq C s^{\frac{2}{7}} \|h_k^2\|_{L^2(\Omega)}^2 \quad (4.4)$$

for some constant C depending only on R_0 and L . Combining (4.1), (4.2), (4.3) and (4.4) yields

$$E_1 \leq (N+1) \sum_{k=0}^N C s^{\frac{2}{7}} \|h_k^2\|_{L^2(\Omega)}^2. \quad (4.5)$$

Similarly we have

$$E_2 \leq (N+1) \sum_{k=0}^N C s^{\frac{2}{7}} \|h_k^1\|_{L^2(\Omega)}^2. \quad (4.6)$$

To finish the proof of the lemma, note that

$$\begin{aligned} & \sum_{k=0}^N (\|h_k^1\|_{L^2(\Omega)}^2 + \|h_k^2\|_{L^2(\Omega)}^2) \\ & \leq s^2 \sum_{k=0}^N (\|\partial_1 u_k - \imath A_k^1 u_k\|_{L^2(\Omega)}^2 + \|\partial_2 u_k - \imath A_k^2 u_k\|_{L^2(\Omega)}^2) \\ & = s^2 \sum_{k=0}^N \|\hat{\nabla}_{\hat{A}_k} u_k\|_{L^2(\Omega)}^2. \end{aligned}$$

Since $\frac{s}{2} \sum_{k=0}^N \|\hat{\nabla}_{\hat{A}_k} u_k\|_{L^2(\Omega)}^2$ is part of the Lawrence-Doniach energy, it follows from Theorem 3.1 that

$$\sum_{k=0}^N (\|h_k^1\|_{L^2(\Omega)}^2 + \|h_k^2\|_{L^2(\Omega)}^2) \leq 2s \mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) \leq 2s M_{\epsilon}(1 + o_{\epsilon,s}(1)).$$

Hence, it follows from this, (4.5) and (4.6) that

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}(\hat{x}, x_3) - \hat{\text{curl}} \hat{A}_n(\hat{x})|^2 d\hat{x} dx_3 \leq E_1 + E_2 \\ & \leq (N+1) C s^{\frac{2}{7}} \sum_{k=0}^N (\|h_k^1\|_{L^2(\Omega)}^2 + \|h_k^2\|_{L^2(\Omega)}^2) \\ & \leq (N+1) C s^{\frac{2}{7}} \cdot s M_{\epsilon}(1 + o_{\epsilon,s}(1)) \leq C s^{\frac{2}{7}} M_{\epsilon} \end{aligned}$$

for all ϵ and s sufficiently small and some constant C depending only on R_0 and L . \square

The above lemma provides the main step in our proof of the lower bound on the minimal Lawrence-Doniach energy.

Theorem 4.2. *Assume $|\ln \epsilon| \ll h_{ex} \ll \frac{1}{\epsilon^2}$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then we have*

$$\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) \geq \frac{|D|}{2} h_{ex} \left(\ln \frac{1}{\epsilon \sqrt{h_{ex}}} \right) (1 - o_\epsilon(1) - Cs^{\frac{1}{7}})$$

for all ϵ and s sufficiently small, where C is a constant depending only on R_0 and L .

Proof. By dropping the nonnegative Josephson coupling term and the square of the L^2 norm of the first two components of $\nabla \times \vec{A} - h_{ex} \vec{e}_3$, it is clear that

$$\begin{aligned} \mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) &\geq s \sum_{n=0}^N \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} (\text{curl} \hat{A} - h_{ex})^2 dx. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} (\text{curl} \hat{A}(x) - h_{ex})^2 dx &\geq \frac{1}{2} \int_D (\text{curl} \hat{A}(x) - h_{ex})^2 dx \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} (\text{curl} \hat{A}(\hat{x}, x_3) - h_{ex})^2 d\hat{x} dx_3 \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} \left[\text{curl}(\hat{A}(\hat{x}, x_3) - \hat{A}_n(\hat{x})) + (\text{curl} \hat{A}_n(\hat{x}) - h_{ex}) \right]^2 d\hat{x} dx_3. \end{aligned}$$

Applying the elementary inequality $(a + b)^2 \geq a^2 - 2|a| \cdot |b|$ yields

$$\begin{aligned} \frac{1}{2} \int_D (\text{curl} \hat{A} - h_{ex})^2 dx &\geq \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} (\text{curl} \hat{A}_n - h_{ex})^2 d\hat{x} dx_3 \\ &\quad - \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\text{curl}(\hat{A} - \hat{A}_n)| \cdot |\text{curl} \hat{A}_n - h_{ex}| d\hat{x} dx_3. \end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned} &\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) \\ &\geq s \sum_{n=0}^{N-1} \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_D (\text{curl} \hat{A} - h_{ex})^2 dx \\ &\geq s \sum_{n=0}^{N-1} \left\{ \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_{\Omega} (\text{curl} \hat{A}_n - h_{ex})^2 d\hat{x} \right\} \\ &\quad - \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\text{curl}(\hat{A} - \hat{A}_n)| \cdot |\text{curl} \hat{A}_n - h_{ex}| d\hat{x} dx_3. \end{aligned} \tag{4.8}$$

It was proved in [2] that $u_n \in H^1(\Omega; \mathbb{C})$ and $\hat{A}_n \in H_{loc}^1(\mathbb{R}^2; \mathbb{R}^2)$ for all $n = 0, 1, \dots, N-1$. Therefore each pair (u_n, \hat{A}_n) is in the admissible set for the minimization of the two-dimensional Ginzburg-Landau

energy F_ϵ , and by Theorem 8.1 in [9], we have, for each $n = 0, 1, \dots, N-1$,

$$\begin{aligned} \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} + \frac{1}{2} \int_{\Omega} (\hat{\text{curl}} \hat{A}_n - h_{ex})^2 d\hat{x} \\ \geq \frac{|\Omega|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - o_\epsilon(1)) \end{aligned} \quad (4.9)$$

as $\epsilon \rightarrow 0$. Using the Cauchy-Schwartz inequality and Hölder's inequality, we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}}(\hat{A} - \hat{A}_n)| \cdot |\hat{\text{curl}} \hat{A}_n - h_{ex}| d\hat{x} dx_3 \\ \leq \left(\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}}(\hat{A} - \hat{A}_n)|^2 d\hat{x} dx_3 \right)^{\frac{1}{2}} \\ \left(\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}_n - h_{ex}|^2 d\hat{x} dx_3 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.10)$$

Since $\hat{\text{curl}} \hat{A}_n - h_{ex} = (\hat{\text{curl}}(\hat{A}_n - \hat{A})) + (\hat{\text{curl}} \hat{A} - h_{ex})$, using the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ yields

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}_n - h_{ex}|^2 d\hat{x} dx_3 \\ \leq 2 \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}}(\hat{A}_n - \hat{A})|^2 d\hat{x} dx_3 + 2 \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A} - h_{ex}|^2 d\hat{x} dx_3. \end{aligned}$$

As a result of this, Lemma 4.1 and Theorem 3.1, we have

$$\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}} \hat{A}_n - h_{ex}|^2 d\hat{x} dx_3 \leq o_{\epsilon,s}(1) M_\epsilon + 4 \cdot M_\epsilon (1 + o_{\epsilon,s}(1)) \leq C_0 \cdot M_\epsilon. \quad (4.11)$$

By (4.10), Lemma 4.1 and (4.11), we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} |\hat{\text{curl}}(\hat{A} - \hat{A}_n)| \cdot |\hat{\text{curl}} \hat{A}_n - h_{ex}| d\hat{x} dx_3 \\ \leq (Cs^{\frac{2}{7}} M_\epsilon)^{\frac{1}{2}} \cdot (C_0 M_\epsilon)^{\frac{1}{2}} = Cs^{\frac{1}{7}} M_\epsilon \end{aligned} \quad (4.12)$$

for some constant C depending only on R_0 and L for all ϵ and s sufficiently small. By (4.7), (4.9), (4.12) and the definition of M_ϵ , i.e., $M_\epsilon = \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}}$, we conclude that

$$\begin{aligned} \mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) &\geq sN \cdot \frac{|\Omega|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - o_\epsilon(1)) - Cs^{\frac{1}{7}} \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} \\ &= \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - o_\epsilon(1)) - Cs^{\frac{1}{7}} \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} \\ &= \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - o_\epsilon(1) - Cs^{\frac{1}{7}}) \end{aligned}$$

for some constant C depending only on R_0 and L . This proves the theorem. \square

By combining Theorems 3.1 and 4.2 we obtain Theorem 1.

Proof of Theorem 2. By (4.7) and (4.12), we see that the leading term in our lower bound of the minimum Lawrence-Doniach energy comes from two terms, since

$$\begin{aligned} \mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) &\geq s \sum_{n=0}^N \int_{\Omega} \left[\frac{1}{2} |\hat{\nabla}_{\hat{A}_n} u_n|^2 + \frac{1}{4\epsilon^2} (1 - |u_n|^2)^2 \right] d\hat{x} \\ &\quad + \frac{1}{2} \int_D (\hat{\text{curl}} \hat{A} - h_{ex})^2 dx \\ &\geq \frac{|D|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - o_{\epsilon}(1) - C s^{\frac{1}{7}}) \end{aligned} \quad (4.13)$$

for some constant C depending only on R_0 and L and for all ϵ and s sufficiently small. As a result of (4.13) and Theorem 3.1, we conclude that

$$\begin{aligned} s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{i \int_{n s}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} &+ \frac{1}{2} \int_{\mathbb{R}^3 \setminus D} |\nabla \times \vec{A} - h_{ex} \vec{e}_3|^2 dx \\ &+ \frac{1}{2} \int_D \left[\left(\frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3} \right)^2 + \left(\frac{\partial A^1}{\partial x_3} - \frac{\partial A^3}{\partial x_1} \right)^2 \right] dx \leq o_{\epsilon,s}(1) M_{\epsilon}. \end{aligned}$$

This proves Theorem 2. □

Corollary 4.3. *Under the assumptions of Theorem 2, we have*

$$|\mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) - \sum_{n=0}^{N-1} s F_{\epsilon}(u_n, \hat{A}_n)| \leq o_{\epsilon,s}(1) \cdot M_{\epsilon}$$

as $(\epsilon, s) \rightarrow (0, 0)$.

Proof. By (4.8) and (4.12), we have

$$\mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) \geq \sum_{n=0}^{N-1} s F_{\epsilon}(u_n, \hat{A}_n) - o_{\epsilon,s}(1) \cdot M_{\epsilon}$$

for all ϵ and s sufficiently small. By the Cauchy-Schwartz inequality, Lemma 4.1, Theorem 3.1 and (4.11), we have

$$\left| \frac{1}{2} \int_D |\hat{\text{curl}} \hat{A} - h_{ex}|^2 dx - \sum_{n=0}^{N-1} \frac{s}{2} \int_{\Omega} |\hat{\text{curl}} \hat{A}_n - h_{ex}|^2 d\hat{x} \right| \leq C s^{\frac{1}{7}} M_{\epsilon}$$

for some constant C depending only on R_0 and L . Combining this with Theorem 2 and the definition of $\mathcal{G}_{LD}^{\epsilon,s}$, we obtain

$$|\mathcal{G}_{LD}^{\epsilon,s}(\{u_n\}_{n=0}^N, \vec{A}) - \sum_{n=0}^{N-1} s F_{\epsilon}(u_n, \hat{A}_n)| \leq o_{\epsilon,s}(1) \cdot M_{\epsilon}.$$

□

Corollary 4.4. *Under the assumptions of Theorem 1, we have*

$$\left\| \frac{\nabla \times \vec{A}}{h_{ex}} - \vec{e}_3 \right\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)} \rightarrow 0,$$

and

$$\frac{1}{N+1} \sum_{n=0}^N \frac{\mu_n}{h_{ex}} \rightarrow d\hat{x} \text{ in } H^{-1}(\Omega)$$

as $(\epsilon, s) \rightarrow (0, 0)$, where μ_n is the vorticity on the n th layer defined as

$$\mu_n = \text{curl}(u_n, \hat{\nabla}_{\hat{A}_n} u_n) + \text{curl} \hat{A}_n.$$

Proof. The convergence of $\frac{\nabla \times \vec{A}}{h_{ex}}$ to \vec{e}_3 in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ follows immediately from Theorem 3.1 and the assumption that $|\ln \epsilon| \ll h_{ex} \ll \frac{1}{\epsilon^2}$.

By the regularity results proved by Bauman and Ko in [2], we know that $(u_n, \hat{\nabla}_{\hat{A}_n} u_n) \in L^2(\Omega; \mathbb{R}^2)$ and $\hat{A}_n \in H^1(\Omega; \mathbb{R}^2)$. Thus

$$\mu_n = \text{curl}(u_n, \hat{\nabla}_{\hat{A}_n} u_n) + \text{curl} \hat{A}_n \in H^{-1}(\Omega),$$

and since $|u_n| \leq 1$, we have

$$\|\mu_n - h_{ex}\|_{H^{-1}(\Omega)}^2 \leq \|\hat{\nabla}_{\hat{A}_n} u_n\|_{L^2(\Omega)}^2 + \|\text{curl} \hat{A}_n - h_{ex}\|_{L^2(\Omega)}^2.$$

Therefore, by Theorem 3.1 and (4.11), we get

$$\begin{aligned} s \sum_{n=0}^N \|\mu_n - h_{ex}\|_{H^{-1}(\Omega)}^2 &\leq s \sum_{n=0}^N \|\hat{\nabla}_{\hat{A}_n} u_n\|_{L^2(\Omega)}^2 + s \sum_{n=0}^N \|\text{curl} \hat{A}_n - h_{ex}\|_{L^2(\Omega)}^2 \\ &= s \sum_{n=0}^N \|\hat{\nabla}_{\hat{A}_n} u_n\|_{L^2(\Omega)}^2 + \sum_{n=0}^N \int_{ns}^{(n+1)s} \int_{\Omega} |\text{curl} \hat{A}_n - h_{ex}|^2 d\hat{x} dx_3 \\ &\leq 2M_\epsilon(1 + o_{\epsilon, s}(1)) + C_0 M_\epsilon \leq C_0 M_\epsilon. \end{aligned}$$

This implies that

$$s \sum_{n=0}^N \left\| \frac{\mu_n}{h_{ex}} - d\hat{x} \right\|_{H^{-1}(\Omega)}^2 \leq \frac{C_0 M_\epsilon}{h_{ex}^2} \rightarrow 0 \quad (4.14)$$

as $(\epsilon, s) \rightarrow (0, 0)$. Note that

$$\begin{aligned} \left\| \frac{1}{N+1} \sum_{n=0}^N \frac{\mu_n}{h_{ex}} - d\hat{x} \right\|_{H^{-1}(\Omega)}^2 &= \left\| \frac{1}{N+1} \sum_{n=0}^N \left(\frac{\mu_n}{h_{ex}} - d\hat{x} \right) \right\|_{H^{-1}(\Omega)}^2 \\ &= \frac{1}{(N+1)^2} \left\| \sum_{n=0}^N \left(\frac{\mu_n}{h_{ex}} - d\hat{x} \right) \right\|_{H^{-1}(\Omega)}^2. \end{aligned}$$

By the Cauchy-Schwartz inequality and (4.14), we obtain

$$\begin{aligned} \left\| \frac{1}{N+1} \sum_{n=0}^N \frac{\mu_n}{h_{ex}} - d\hat{x} \right\|_{H^{-1}(\Omega)}^2 &\leq \frac{N+1}{(N+1)^2} \sum_{n=0}^N \left\| \frac{\mu_n}{h_{ex}} - d\hat{x} \right\|_{H^{-1}(\Omega)}^2 \\ &= \frac{1}{s(N+1)} \cdot s \sum_{n=0}^N \left\| \frac{\mu_n}{h_{ex}} - d\hat{x} \right\|_{H^{-1}(\Omega)}^2 \rightarrow 0 \end{aligned}$$

as $(\epsilon, s) \rightarrow (0, 0)$, since $sN = L$ is the height of the domain D which is fixed. \square

5 Comparison results

In this section we prove a comparison result between the minimum Lawrence-Doniach energy and the minimum three-dimensional anisotropic Ginzburg-Landau energy under the assumption that $s \leq C_0 \epsilon$ for some constant C_0 independent of ϵ , s , Ω , L , D and R_0 for all ϵ and s sufficiently small. Recall the definition of the anisotropic Ginzburg-Landau energy $\mathcal{G}_{AGL}^\epsilon$ given in (1.8) in the introduction. Direct calculations show that $\mathcal{G}_{AGL}^\epsilon$ is invariant under the gauge transformation

$$\begin{cases} \xi(x) = \psi(x)e^{ig(x)} \text{ in } \Omega, \\ \vec{B} = \vec{A} + \nabla g \text{ in } \mathbb{R}^3 \end{cases}$$

for some $g \in H_{loc}^2(\mathbb{R}^3)$. Recall the rescaling formulas (2.4) for the anisotropic Ginzburg-Landau energies, from which we may translate estimates from [3], [2] and [11] to our scaling. As pointed out in [2], every minimizer $(\psi, \vec{A}) \in H^1(D; \mathbb{C}) \times E$ of $\mathcal{G}_{AGL}^\epsilon$ is gauge equivalent to another pair in $H^1(D; \mathbb{C}) \times K$, where the spaces E and K are defined in (1.2) and (1.4) respectively. The space $H^1(D; \mathbb{C}) \times K$ fixes a ‘‘Coulomb gauge’’ for (ψ, \vec{A}) as in the study of the Lawrence-Doniach energy. Our goal of this section is to prove Theorem 3. First we prove several lemmas that will be used for an upper bound on the minimal three-dimensional anisotropic Ginzburg-Landau energy.

Lemma 5.1. *Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$. Then*

$$\|A^3\|_{L^6(D)}^2 \leq C_0 \cdot M_\epsilon$$

for all ϵ and s sufficiently small and some constant C_0 independent of ϵ , s , Ω , L , D and R_0 .

Proof. Since $a^3 = 0$, by (2.1), (2.2) and Theorem 3.1 we have

$$\begin{aligned} \|A^3\|_{L^6(D)}^2 &= \|A^3 - h_{ex}a^3\|_{L^6(D)}^2 \leq \|\vec{A} - h_{ex}\vec{a}\|_{L^6(D)}^2 \\ &\leq 4\|\vec{A} - h_{ex}\vec{a}\|_{H^1(\mathbb{R}^3)}^2 = 4 \int_{\mathbb{R}^3} |\nabla \times (\vec{A} - h_{ex}\vec{a})|^2 dx \\ &\leq 8\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}) \leq C_0 \cdot M_\epsilon \end{aligned}$$

for C_0 as described above and all ϵ and s sufficiently small. □

Lemma 5.2. *Under the assumptions above and in addition assuming $s \leq C_0 \epsilon$, we have*

$$\sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{s} |u_{n+1} - u_n|^4 d\hat{x} \leq o_\epsilon(1) M_\epsilon$$

as $\epsilon \rightarrow 0$.

Proof. We write

$$|u_{n+1} - u_n| = |(u_{n+1} - u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3}) + (u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3} - u_n)|.$$

Using the triangle inequality and $(a + b)^4 \leq 2^4(a^4 + b^4)$ for $a, b > 0$, we obtain

$$\sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{s} |u_{n+1} - u_n|^4 d\hat{x} \leq J_1 + J_2,$$

where

$$J_1 = \sum_{n=0}^{N-1} \int_{\Omega} \frac{2^4}{s} |u_{n+1} - u_n e^{i \int_{ns}^{(n+1)s} A^3 dx_3}|^4 d\hat{x}$$

and

$$J_2 = \sum_{n=0}^{N-1} \int_{\Omega} \frac{2^4}{s} |u_n e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3} - u_n|^4 d\hat{x}.$$

Note that, since $|u_{n+1}| \leq 1$ and $|u_n e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}| = |u_n| \leq 1$,

$$J_1 \leq \sum_{n=0}^{N-1} \int_{\Omega} \frac{2^4}{s} \cdot 2^2 |u_{n+1} - u_n e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x}.$$

By Theorem 2 and our assumption that λ is fixed, it follows that $J_1 \leq o_{\epsilon,s}(1)M_{\epsilon}$.

For J_2 , following an idea in [11] and Hölder's inequality, we have

$$\begin{aligned} J_2 &\leq \frac{2^4}{s} \sum_{n=0}^{N-1} \int_{\Omega} \left| \int_{ns}^{(n+1)s} A^3 dx_3 \right|^4 d\hat{x} \leq \frac{2^4}{s} \sum_{n=0}^{N-1} \int_{\Omega} s^3 \int_{ns}^{(n+1)s} (A^3)^4 dx_3 d\hat{x} \\ &= 2^4 s^2 \int_D (A^3)^4 dx \leq 2^4 s^2 \left(\int_D (A^3)^6 dx \right)^{\frac{2}{3}} |D|^{\frac{1}{3}} = 2^4 s^2 |D|^{\frac{1}{3}} \|A^3\|_{L^6(D)}^4. \end{aligned}$$

Since $s \leq C_0 \epsilon$ and $\epsilon^2 M_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$, these imply that $s^2 M_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore, Lemma 5.1 implies that $J_2 \leq o_{\epsilon,s}(1)M_{\epsilon}$ and hence the lemma is proved. \square

In the following we will need several additional lemmas which follow from Lemma 5.1, Lemma 5.2 and calculations from the proof of Lemma 5.5 in [3] (after appropriate rescaling). We remark that Lemma 5.2 is a stronger estimate than the analogous estimate in [3] and [11] (in which ϵ was fixed), and it is a key ingredient in our proof of Theorem 3.

Lemma 5.3. *Assume $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ and $s \leq C_0 \epsilon$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon,s}$. Define $\psi \in H^1(D; \mathbb{C})$ to be*

$$\psi(\hat{x}, x_3) \equiv \sum_{n=0}^{N-1} [(1-t_n)u_n(\hat{x}) + t_n u_{n+1}(\hat{x})] \chi_{[ns, (n+1)s]}(x_3),$$

where $t_n = \frac{x_3 - ns}{s}$. Then we have

$$\begin{aligned} &\frac{1}{2} \int_D \frac{1}{\lambda^2} \left| \left(\frac{\partial}{\partial x_3} - \imath A^3 \right) \psi \right|^2 dx \\ &\leq s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} + o_{\epsilon}(1)M_{\epsilon} \end{aligned} \tag{5.1}$$

as $\epsilon \rightarrow 0$.

Proof. By the definition of ψ , we have

$$\begin{aligned} &\frac{1}{2} \int_D \frac{1}{\lambda^2} \left| \left(\frac{\partial}{\partial x_3} - \imath A^3 \right) \psi \right|^2 dx - s \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{2\lambda^2 s^2} |u_{n+1} - u_n e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} \\ &= \frac{1}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} \left\{ \int_{ns}^{(n+1)s} \left| (u_{n+1} - u_n)(1 - \imath s t_n A^3) - \imath s A^3 u_n \right|^2 dx_3 \right. \\ &\quad \left. - s \left| (u_{n+1} - u_n) + u_n (1 - e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}) \right|^2 \right\} d\hat{x} \\ &= (\tilde{R}_{11} - \tilde{R}_{21}) + (\tilde{R}_{12} - \tilde{R}_{22}) + (\tilde{R}_{13} - \tilde{R}_{23}), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned}\tilde{R}_{11} - \tilde{R}_{21} &= \frac{1}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} \left[\int_{ns}^{(n+1)s} |(u_{n+1} - u_n)(1 - \imath st_n A^3)|^2 dx_3 - s|u_{n+1} - u_n|^2 \right] d\hat{x}, \\ \tilde{R}_{12} - \tilde{R}_{22} &= \frac{1}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} 2\Re \left[\int_{ns}^{(n+1)s} (u_{n+1} - u_n)(1 - \imath st_n A^3) \imath s A^3 \bar{u}_n dx_3 \right. \\ &\quad \left. - s(u_{n+1} - u_n) \bar{u}_n (1 - e^{-\imath \int_{ns}^{(n+1)s} A^3 dx_3}) \right] d\hat{x},\end{aligned}$$

and

$$\tilde{R}_{13} - \tilde{R}_{23} = \frac{1}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} \left[\int_{ns}^{(n+1)s} s^2 (A^3)^2 |u_n|^2 dx_3 - s|u_n|^2 \cdot |1 - e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}|^2 \right] d\hat{x}.$$

Here we use the splitting in the proof of Lemma 5.5 in [3]. Each $\tilde{R}_{1j} - \tilde{R}_{2j}$ corresponds to the quantity $R_{1j} - R_{2j}$ in the proof of Lemma 5.5 in [3] for $j = 1, 2, 3$.

Note that $|1 - \imath st_n A^3|^2 = 1 + (st_n A^3)^2$ and, by the definition of t_n , $st_n = x_3 - ns$. Therefore we have

$$\begin{aligned}\tilde{R}_{11} - \tilde{R}_{21} &= \frac{1}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} \left[\int_{ns}^{(n+1)s} |u_{n+1} - u_n|^2 (1 + (x_3 - ns)^2 (A^3)^2) dx_3 - s|u_{n+1} - u_n|^2 \right] d\hat{x}.\end{aligned}$$

From the estimates for $|R_{11} - R_{21}|$ in the proof of Lemma 5.5 in [3] and using the rescaling relations (2.3) and (2.4) (in particular, note that $\psi_n = u_n$ and $\kappa A_z = A^3$), we have

$$\begin{aligned}& \left| \frac{1}{s^2} \sum_{n=0}^{N-1} \int_{\Omega} \left[\int_{ns}^{(n+1)s} |u_{n+1} - u_n|^2 (1 + (x_3 - ns)^2 (A^3)^2) dx_3 - s|u_{n+1} - u_n|^2 \right] d\hat{x} \right| \\ & \leq 2s^{\frac{1}{2}} \left[s \sum_{n=0}^{N-1} \int_{\Omega} \left(\frac{|u_{n+1} - u_n|}{s} \right)^2 d\hat{x} \right]^{\frac{1}{2}} \cdot s^{\frac{1}{2}} \cdot \left[\int_0^L \int_{\Omega} |A^3|^4 d\hat{x} dx_3 \right]^{\frac{1}{2}}.\end{aligned}\tag{5.3}$$

It follows from (5.3) that

$$\tilde{R}_{11} - \tilde{R}_{21} \leq \frac{s}{\lambda^2} \left(\sum_{n=0}^{N-1} \frac{1}{s} \int_{\Omega} |u_{n+1} - u_n|^2 d\hat{x} \right)^{\frac{1}{2}} \cdot \left(\int_D (A^3)^4 dx \right)^{\frac{1}{2}}.\tag{5.4}$$

Using similar calculations as in the proof of Lemma 5.2, it is not hard to show that

$$\sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{s} |u_{n+1} - u_n|^2 d\hat{x} \leq C \cdot M_{\epsilon}$$

for some constant C depending only on $|D|$ for all ϵ sufficiently small. Then applying Hölder's inequality to $\int_D (A^3)^4 dx$ as in the proof of Lemma 5.2 and using Lemma 5.1, we deduce from (5.4) that

$$\tilde{R}_{11} - \tilde{R}_{21} \leq Cs \cdot \sqrt{M_{\epsilon}} \cdot M_{\epsilon}$$

for some constant C depending only on $|D|$. From the proof of Lemma 5.2, $s\sqrt{M_{\epsilon}} \rightarrow 0$ as $\epsilon \rightarrow 0$. It follows that

$$\tilde{R}_{11} - \tilde{R}_{21} \leq o_{\epsilon}(1) M_{\epsilon}.\tag{5.5}$$

Next, since $(1 - \imath st_n A^3) \imath A^3 = \imath A^3 + (x_3 - ns)(A^3)^2$, we see that

$$\begin{aligned} \tilde{R}_{12} - \tilde{R}_{22} = & \frac{s^2}{2\lambda^2 s^2} \sum_{n=0}^{N-1} \int_{\Omega} 2\Re \left[\int_{ns}^{(n+1)s} \left(\frac{u_{n+1} - u_n}{s} \right) \bar{u}_n (\imath A^3 + (x_3 - ns)(A^3)^2) dx_3 \right. \\ & \left. - \left(\frac{u_{n+1} - u_n}{s} \right) \bar{u}_n (1 - e^{-\imath \int_{ns}^{(n+1)s} A^3 dx_3}) \right] d\hat{x}. \end{aligned}$$

Using the estimates for $|R_{12} - R_{22}|$ in the proof of Lemma 5.5 in [3], we have

$$\begin{aligned} & 2 \left| \Re \left\{ \sum_{n=0}^{N-1} \int_{\Omega} \left(\frac{u_{n+1} - u_n}{s} \right) \bar{u}_n \left(\int_{ns}^{(n+1)s} [\imath A^3 + (x_3 - ns)(A^3)^2] dx_3 \right) d\hat{x} \right\} \right. \\ & \left. - \Re \left\{ \sum_{n=0}^{N-1} \int_{\Omega} \left(\frac{u_{n+1} - u_n}{s} \right) \bar{u}_n (1 - e^{-\imath \int_{ns}^{(n+1)s} A^3 dx_3}) d\hat{x} \right\} \right| \\ & \leq 4 \sum_{n=0}^{N-1} \int_{\Omega} \left(|u_{n+1} - u_n| \int_{ns}^{(n+1)s} |A^3|^2 dx_3 \right) d\hat{x}. \end{aligned} \quad (5.6)$$

Then (5.6) yields

$$\tilde{R}_{12} - \tilde{R}_{22} \leq \frac{2}{\lambda^2} \sum_{n=0}^{N-1} \int_{\Omega} |u_{n+1} - u_n| \left(\int_{ns}^{(n+1)s} |A^3|^2 dx_3 \right) d\hat{x}. \quad (5.7)$$

Applying Hölder's inequality and the Cauchy-Schwartz inequality reduces the estimate for $\tilde{R}_{12} - \tilde{R}_{22}$ in (5.7) to that for $\tilde{R}_{11} - \tilde{R}_{21}$ in (5.4), and thus

$$\tilde{R}_{12} - \tilde{R}_{22} \leq o_{\epsilon}(1) M_{\epsilon}. \quad (5.8)$$

Translating the estimates for $|R_{13} - R_{23}|$ in [3] gives

$$\begin{aligned} & \left| \sum_{n=0}^{N-1} \int_{\Omega} [|u_n|^2 \cdot \int_{ns}^{(n+1)s} |A^3|^2 dx_3] d\hat{x} - \frac{1}{s} \sum_{n=0}^{N-1} \int_{\Omega} |u_n|^2 \cdot |1 - e^{-\imath \int_{ns}^{(n+1)s} A^3 dx_3}|^2 d\hat{x} \right| \\ & \leq \frac{s^2}{12} \int_{\Omega} \int_0^L |A^3|^4 dx_3 d\hat{x} + 2s \left\| \frac{\partial A^3}{\partial x_3} \right\|_{L^2(D)} \cdot \|A^3\|_{L^2(D)}. \end{aligned} \quad (5.9)$$

Observing $|1 - e^{-\imath \int_{ns}^{(n+1)s} A^3 dx_3}| = |1 - e^{\imath \int_{ns}^{(n+1)s} A^3 dx_3}|$, it follows from (5.9) that

$$\tilde{R}_{13} - \tilde{R}_{23} \leq \frac{1}{2\lambda^2} \frac{s^2}{12} \int_D (A^3)^4 dx + \frac{s}{\lambda^2} \left\| \frac{\partial A^3}{\partial x_3} \right\|_{L^2(D)} \cdot \|A^3\|_{L^2(D)}.$$

From (2.2) we have

$$\begin{aligned} \int_D \left| \frac{\partial A^3}{\partial x_3} \right|^2 dx & \leq \int_{\mathbb{R}^3} |\nabla(\vec{A} - h_{ex} \vec{a})|^2 dx \\ & = \int_{\mathbb{R}^3} |\nabla \times (\vec{A} - h_{ex} \vec{a})|^2 dx \leq 2\mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A}). \end{aligned}$$

By Hölder's inequality and Lemma 5.1 we have

$$\|A^3\|_{L^2(D)}^2 \leq C \|A^3\|_{L^6(D)}^2 \leq C \cdot M_{\epsilon}$$

for some constant C depending only on $|D|$ for all ϵ sufficiently small, and therefore

$$\frac{s}{\lambda^2} \left\| \frac{\partial A^3}{\partial x_3} \right\|_{L^2(D)} \|A^3\|_{L^2(D)} \leq CsM_\epsilon = o_\epsilon(1)M_\epsilon.$$

It is easy to see that

$$\frac{1}{2\lambda^2} \frac{s^2}{12} \int_D (A^3)^4 dx \leq o_\epsilon(1)M_\epsilon$$

and therefore

$$\tilde{R}_{13} - \tilde{R}_{23} \leq o_\epsilon(1)M_\epsilon. \quad (5.10)$$

Combining the estimates (5.5), (5.8) and (5.10) and using (5.2), we obtain (5.1). \square

Lemma 5.4. *Assuming the hypotheses of Lemma 5.3, we have*

$$\frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx \leq \frac{s}{2} \sum_{n=0}^N \int_\Omega |\hat{\nabla}_{\hat{A}_n} u_n|^2 d\hat{x} + o_\epsilon(1)M_\epsilon \quad (5.11)$$

as $\epsilon \rightarrow 0$.

Proof. We calculate $\frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx$:

$$\begin{aligned} \frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx &= \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left| \hat{\nabla}_{\hat{A}} [(1-t_n)u_n(\hat{x}) + t_n u_{n+1}(\hat{x})] \right|^2 d\hat{x} dx_3 \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left| (1-t_n) \hat{\nabla}_{\hat{A}} u_n(\hat{x}) + t_n \hat{\nabla}_{\hat{A}} u_{n+1}(\hat{x}) \right|^2 d\hat{x} dx_3. \end{aligned}$$

Using the triangle inequality and the convexity of the function x^2 , we obtain

$$\begin{aligned} \frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx &\leq \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left[(1-t_n) |\hat{\nabla}_{\hat{A}} u_n(\hat{x})|^2 + t_n |\hat{\nabla}_{\hat{A}} u_{n+1}(\hat{x})|^2 \right] d\hat{x} dx_3 \\ &= \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left[(1-t_n) \left| \hat{\nabla}_{\hat{A}_n} u_n(\hat{x}) + \imath(\hat{A}_n - \hat{A}) u_n \right|^2 \right] d\hat{x} dx_3 \\ &\quad + \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left[t_n \left| \hat{\nabla}_{\hat{A}_{n+1}} u_{n+1}(\hat{x}) + \imath(\hat{A}_{n+1} - \hat{A}) u_{n+1} \right|^2 \right] d\hat{x} dx_3. \end{aligned}$$

Expanding the above quadratic terms and applying Young's inequality $ab \leq \frac{sa^2}{2} + \frac{1}{2s}b^2$ to the cross product terms, we obtain

$$\begin{aligned} \frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx &\leq \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left[(1-t_n)(1+s) |\hat{\nabla}_{\hat{A}_n} u_n|^2 \right. \\ &\quad \left. + (1-t_n)(1+s^{-1}) |\hat{A}_n - \hat{A}|^2 \right] d\hat{x} dx_3 \\ &\quad + \frac{1}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_\Omega \left[t_n(1+s) |\hat{\nabla}_{\hat{A}_{n+1}} u_{n+1}|^2 \right. \\ &\quad \left. + t_n(1+s^{-1}) |\hat{A}_{n+1} - \hat{A}|^2 \right] d\hat{x} dx_3. \end{aligned} \quad (5.12)$$

Note that $\int_{ns}^{(n+1)s} t_n dx_3 = \int_{ns}^{(n+1)s} (1 - t_n) dx_3 = \frac{s}{2}$. Using $\sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x}$ as an upper bound for both $\sum_{n=0}^{N-1} \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x}$ and $\sum_{n=0}^{N-1} \int_{\Omega} \frac{(1-|u_{n+1}|^2)^2}{4\epsilon^2} d\hat{x}$, we deduce from (5.12) that

$$\frac{1}{2} \int_D |\hat{\nabla}_{\hat{A}} \psi|^2 dx \leq \frac{s}{2} \sum_{n=0}^N \int_{\Omega} |\hat{\nabla}_{\hat{A}_n} u_n|^2 d\hat{x} + R_4 \quad (5.13)$$

where

$$\begin{aligned} R_4 &= \frac{1+s^{-1}}{2} \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} \left[(1-t_n) |\hat{A}_n - \hat{A}|^2 + t_n |\hat{A}_{n+1} - \hat{A}|^2 \right] d\hat{x} dx_3 \\ &\quad + \frac{s^2}{2} \sum_{n=0}^N \int_{\Omega} |\hat{\nabla}_{\hat{A}_n} u_n|^2 d\hat{x}. \end{aligned}$$

Using the idea in [11], we have

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} (1-t_n) |\hat{A}_n - \hat{A}|^2 d\hat{x} dx_3 \\ &\leq \sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} (1-t_n) \left(\int_{ns}^{(n+1)s} \left| \frac{\partial \hat{A}}{\partial x_3} \right| dx_3 \right)^2 d\hat{x} dx_3 \\ &\leq \frac{s}{2} \cdot s \sum_{n=0}^{N-1} \int_{\Omega} \int_{ns}^{(n+1)s} \left| \frac{\partial \hat{A}}{\partial x_3} \right|^2 dx_3 d\hat{x} = \frac{s^2}{2} \int_D \left| \frac{\partial}{\partial x_3} (\hat{A} - h_{ex} \hat{a}) \right|^2 dx. \end{aligned}$$

It follows from (2.2) that

$$\int_D \left| \frac{\partial}{\partial x_3} (\hat{A} - h_{ex} \hat{a}) \right|^2 dx \leq \int_{\mathbb{R}^3} |\nabla (\vec{A} - h_{ex} \vec{a})|^2 dx = \int_{\mathbb{R}^3} |\nabla \times (\vec{A} - h_{ex} \vec{a})|^2 dx$$

and by Theorem 3.1,

$$\sum_{n=0}^{N-1} \int_{ns}^{(n+1)s} \int_{\Omega} (1-t_n) |\hat{A}_n - \hat{A}|^2 d\hat{x} dx_3 \leq s^2 M_{\epsilon} (1 + o_{\epsilon}(1)).$$

From this it is clear that $R_4 \leq o_{\epsilon}(1) M_{\epsilon}$. Thus (5.11) follows from (5.13). \square

Lemma 5.5. *Assuming the hypotheses of Lemma 5.3, we have*

$$\int_D \frac{(1-|\psi|^2)^2}{4\epsilon^2} dx \leq s \sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x} + o_{\epsilon}(1) M_{\epsilon} \quad (5.14)$$

as $\epsilon \rightarrow 0$.

Proof. By the definition of ψ and direct calculations we obtain, for $(\hat{x}, x_3) \in \Omega \times [ns, (n+1)s]$,

$$1 - |\psi|^2 = (1 - t_n)(1 - |u_n|^2) + t_n(1 - |u_{n+1}|^2) + t_n(1 - t_n)|u_n - u_{n+1}|^2.$$

Explicit calculations using the definition of t_n gives

$$\begin{aligned} \int_D \frac{(1-|\psi|^2)^2}{4\epsilon^2} dx &= \frac{1}{4\epsilon^2} \frac{s}{15} \sum_{n=0}^{N-1} \int_{\Omega} \left[5(1-|u_n|^2)^2 + 5(1-|u_{n+1}|^2)^2 \right. \\ &\quad \left. + 5(1-|u_n|^2)(1-|u_{n+1}|^2) + \frac{1}{2}|u_n - u_{n+1}|^4 \right. \\ &\quad \left. + \frac{5}{2}(1-|u_n|^2)|u_n - u_{n+1}|^2 + \frac{5}{2}(1-|u_{n+1}|^2)|u_n - u_{n+1}|^2 \right] d\hat{x}. \end{aligned}$$

Applying Young's inequality $ab \leq \delta(\epsilon)a^2 + \frac{1}{\delta(\epsilon)}b^2$ to $(1-|u_n|^2)|u_n - u_{n+1}|^2$ and $(1-|u_{n+1}|^2)|u_n - u_{n+1}|^2$ for some $\delta(\epsilon) > 0$ depending only on ϵ and to be chosen later, and applying the elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ to $(1-|u_n|^2)(1-|u_{n+1}|^2)$, we obtain

$$\begin{aligned} \int_D \frac{(1-|\psi|^2)^2}{4\epsilon^2} dx &\leq \frac{1}{4\epsilon^2} \frac{s}{15} \sum_{n=0}^{N-1} \int_{\Omega} \left[\frac{15}{2} \left((1-|u_n|^2)^2 + (1-|u_{n+1}|^2)^2 \right) \right. \\ &\quad \left. + \frac{5\delta(\epsilon)}{2} \left((1-|u_n|^2)^2 + (1-|u_{n+1}|^2)^2 \right) \right] d\hat{x} \\ &\quad + \frac{1}{4\epsilon^2} \frac{s}{15} \sum_{n=0}^{N-1} \int_{\Omega} \left(\frac{1}{2} + \frac{5}{\delta(\epsilon)} \right) |u_n - u_{n+1}|^4 d\hat{x} \\ &= R_{51} + R_{52}. \end{aligned} \tag{5.15}$$

Note that

$$R_{51} \leq s \sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x} + \frac{\delta(\epsilon)}{3} \cdot s \sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x}$$

where we have used $\sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x}$ to bound both

$$\sum_{n=0}^{N-1} \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x}$$

and

$$\sum_{n=0}^{N-1} \int_{\Omega} \frac{(1-|u_{n+1}|^2)^2}{4\epsilon^2} d\hat{x}$$

from above again. Also note that, by the assumption $s \leq C_0\epsilon$ and Lemma 5.2, we have

$$R_{52} \leq C_0 \left(\frac{1}{2} + \frac{5}{\delta(\epsilon)} \right) \sum_{n=0}^{N-1} \int_{\Omega} \frac{1}{s} |u_n - u_{n+1}|^4 d\hat{x} \leq C_0 \left(\frac{1}{2} + \frac{5}{\delta(\epsilon)} \right) \cdot c(\epsilon) M_{\epsilon},$$

where $c(\epsilon)$ depends only on ϵ for all ϵ sufficiently small and satisfies $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Now we choose $\delta(\epsilon)$ to satisfy $\delta(\epsilon) \rightarrow 0$ and $\frac{c(\epsilon)}{\delta(\epsilon)} \rightarrow 0$ as $\epsilon \rightarrow 0$. (For example, we may choose $\delta(\epsilon) = \sqrt{c(\epsilon)}$.) Then we obtain

$$R_{51} \leq s \sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x} + o_{\epsilon}(1) \cdot s \sum_{n=0}^N \int_{\Omega} \frac{(1-|u_n|^2)^2}{4\epsilon^2} d\hat{x} \tag{5.16}$$

and

$$R_{52} \leq o_\epsilon(1)M_\epsilon \quad (5.17)$$

as $\epsilon \rightarrow 0$. Combining (5.16) and (5.17) with (5.15) and using Theorem 3.1, we obtain (5.14). \square

Theorem 5.6. *Assume $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ and $s \leq C_0 \epsilon$ as $\epsilon \rightarrow 0$. Let $(\{u_n\}_{n=0}^N, \vec{A}) \in [H^1(\Omega; \mathbb{C})]^{N+1} \times K$ be a minimizer of $\mathcal{G}_{LD}^{\epsilon, s}$ and $(\zeta, \vec{B}) \in H^1(D; \mathbb{C}) \times K$ be a minimizer of $\mathcal{G}_{AGL}^\epsilon$. We have*

$$\mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) \leq \mathcal{G}_{LD}^{\epsilon, s}(\{u_n\}_{n=0}^N, \vec{A})(1 + o_\epsilon(1)) \quad (5.18)$$

as $\epsilon \rightarrow 0$.

Proof. Let ψ be defined as in Lemma 5.3 and let $\tilde{A} \equiv \vec{A}$. We show that $\mathcal{G}_{AGL}^\epsilon(\psi, \tilde{A})$ satisfies (5.18). Since we are taking the same magnetic potential \vec{A} for $\mathcal{G}_{LD}^{\epsilon, s}$ and $\mathcal{G}_{AGL}^\epsilon$, the magnetic terms in the two energies are equal. Then the comparison result (5.18) follows from (5.1), (5.11) and (5.14). \square

Next we prove a lower bound for the minimal anisotropic Ginzburg-Landau energy $\mathcal{G}_{AGL}^\epsilon$. The proof is based on a slicing method in which we use the lower bound for the two-dimensional Ginzburg-Landau energy proved by Sandier and Serfaty in [9] on each cross section $\Omega \times \{x_3\}$, and integrate over x_3 to get the desired lower bound for $\mathcal{G}_{AGL}^\epsilon$.

Theorem 5.7. *Assume $|\ln \epsilon| \ll h_{ex} \ll \epsilon^{-2}$ as $\epsilon \rightarrow 0$. Let $(\zeta, \vec{B}) \in H^1(D; \mathbb{C}) \times E$ be a minimizer of $\mathcal{G}_{AGL}^\epsilon$. Then we have*

$$\mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) \geq M_\epsilon(1 - o_\epsilon(1))$$

as $\epsilon \rightarrow 0$.

Proof. Dropping appropriate terms in $\mathcal{G}_{AGL}^\epsilon$, we see that

$$\begin{aligned} \mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) &\geq \int_D \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}} \zeta|^2 + \frac{(1 - |\zeta|^2)^2}{4\epsilon^2} \right] dx + \frac{1}{2} \int_D (\text{curl} \hat{B} - h_{ex})^2 dx \\ &= \int_0^L \left\{ \int_\Omega \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}} \zeta|^2 + \frac{(1 - |\zeta|^2)^2}{4\epsilon^2} \right] d\hat{x} + \frac{1}{2} \int_\Omega (\text{curl} \hat{B} - h_{ex})^2 d\hat{x} \right\} dx_3. \end{aligned}$$

Since $\zeta \in H^1(D; \mathbb{C})$ and $\hat{B} \in H_{loc}^1(\mathbb{R}^3; \mathbb{R}^2)$, we have that for almost every $x_3 \in [0, L]$, $\zeta(\cdot, x_3) \in H^1(\Omega \times \{x_3\}; \mathbb{C})$ and $\hat{B}(\cdot, x_3) \in H^1(\Omega \times \{x_3\}; \mathbb{R}^2)$. By Theorem 8.1 in [9], we have

$$\min_{(v, b) \in H^1 \times H^1} F_\epsilon(v, b) \geq \frac{|\Omega|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - c(\epsilon))$$

for some constant $c(\epsilon)$ depending only on ϵ such that $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Thus, for almost every $x_3 \in [0, L]$, we have

$$\int_\Omega \left[\frac{1}{2} |\hat{\nabla}_{\hat{B}} \zeta|^2 + \frac{(1 - |\zeta|^2)^2}{4\epsilon^2} \right] d\hat{x} + \frac{1}{2} \int_\Omega (\text{curl} \hat{B} - h_{ex})^2 d\hat{x} \geq \frac{|\Omega|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - c(\epsilon)).$$

Integrating from 0 to L , we obtain

$$\mathcal{G}_{AGL}^\epsilon(\zeta, \vec{B}) \geq L \frac{|\Omega|}{2} h_{ex} \ln \frac{1}{\epsilon \sqrt{h_{ex}}} (1 - c(\epsilon)) = M_\epsilon(1 - c(\epsilon))$$

for all ϵ sufficiently small. The theorem follows from this and the property of $c(\epsilon)$ mentioned above. \square

Theorem 3 follows by combining Theorems 5.6, 5.7, 3.1 and 4.2.

References

- [1] S. ALAMA, L. BRONSARD AND E. SANDIER, *On the Lawrence-Doniach model of superconductivity: magnetic fields parallel to the axes*, J. Eur. Math. Soc., 14 (2012), pp. 1825-1857.
- [2] P. BAUMAN AND Y. KO, *Analysis of solutions to the Lawrence-Doniach system for layered superconductors*, SIAM J. Math. Anal., 37 (2005), pp. 914-940.
- [3] S. J. CHAPMAN, Q. DU AND M. D. GUNZBURGER, *On the Lawrence-Doniach and anisotropic Ginzburg-Landau models for layered superconductors*, SIAM J. Appl. Math., 55 (1995), pp. 156-174.
- [4] E. B. FABES, M. JODEIT JR AND N. M. RIVIÈRE, *Potential techniques for boundary value problems on C^1 -domains*, Acta Math., 141 (1978), pp. 165-186.
- [5] D. GILBARG AND N. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, reprint of the 1998 ed., Springer-Verlag, Berlin, 2001.
- [6] T. GIORGI AND D. PHILLIPS, *The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model*, SIAM Review, 44 (2002), pp. 237-256.
- [7] Y. IYE, *How anisotropic are the cuprate high T_c superconductors?*, Comments Cond. Mat. Phys., 16 (1992), pp. 89-111.
- [8] E. SANDIER AND S. SERFATY, *On the energy of type-II superconductors in the mixed phase*, Rev. Math. Phys., 12 (2000), pp. 1219-1257.
- [9] E. SANDIER AND S. SERFATY, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and Their Applications 70, Birkhäuser, Boston, 2007.
- [10] G. VERCHOTA, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), pp. 572-611.
- [11] C. YUAN, *Gamma-convergence of the Lawrence-Doniach energy to a three-dimensional anisotropic Ginzburg-Landau energy*, Ph.D. thesis, Purdue University, West Lafayette, IN, 2008.